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Two Body Dirac Equations and Nucleon Nucleon Scattering Phase Shift Analysis

Bin Liu

University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by Bin Liu entitled "Two Body Dirac Equations and Nucleon Nucleon Scattering Phase Shift Analysis." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Physics.

Dr. Horace Crater, Major Professor

We have read this dissertation and recommend its acceptance:

Dr. Lloyd Davis, Dr. James Lewis, Dr. Chris Parigger, Dr. John Steinhoff

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Dr. Anne Mayhew

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Two Body Dirac Equations and Nucleon Nucleon Scattering Phase Shift Analysis

A Dissertation

Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Bin Liu

December 2001

Dedication

Dedicated to my parents and my family.

Acknowledgments

I would like to thank Dr. Horace Crater who has been my mentor since I arrived at the University, for his patience and guidance to my academic achievements. I am also grateful to other members of my dissertation committee, Dr. Lloyd Davis, Dr. James Lewis, Dr. Chris Parigger and Dr. John Steinhoff, for their advice and assistance over the past five years.

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Abstract

In this dissertation, the nucleon-nucleon interaction is investigated by using the meson exchange model and the two body Dirac equations of constraint dynamics. This approach to the two body problem has been successfully tested for QED and QCD relativistic bound states. An important question we wish to address is whether or not this approach is also valid in the two body nucleon-nucleon scattering problem. This test involves a number of related problems.

First we must reduce our two body Dirac equations exactly to a Schrödinger-like equation. This can be done without making any assumptions or approximations and unlike other relativistic approaches these equations have effective potentials that are local. We then develop a matrix scale transformation that successfully removes first derivative terms that appear naturally in those Schrödinger-like equations because they are inherent in the our two body Dirac equations. This removal is important since it then allows us to use techniques to solve our two body Dirac equations that have been already developed for Schrödinger-like systems in nonrelativistic quantum mechanics.

We use nine mesons in our two body Dirac equations to fit the experimental scattering phase shifts for $n-p$ scattering. The data involves seven angular momentum states including the singlet states ${}^1S_0$, ${}^1P_1$, ${}^1D_2$ and the triplet states ${}^3P_0$, ${}^3P_1$, ${}^3S_1$, ${}^3D_1$. Two models that we have tested give us a fairly good fit. The nucleon-nucleon potentials that we use are also called the semi-phenomenological potentials due to the incorporating of

the meson exchange model into the invariant potentials appearing in our Dirac equation. Our approach gives the nucleon-nucleon interaction a physical meaning beyond just the curve fitting which a purely phenomenological potentials provides.

The parameters obtained by fitting the $n-p$ experimental scattering phase shift give a fairly good prediction for most of the $p-p$ experimental scattering phase shift(for states of singlet ${}^1S_0$, ${}^1D_2$ and triplet ${}^3P_0$, ${}^3P_1$). This means that the two body Dirac equations of constraint dynamics show promise in describing the nucleon-nucleon interaction. We outline generalizations of the meson exchange model for the invariant potentials that may possibly improve the fit.

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Chapter 1

Introduction

Nuclear physics is the study of the structure of the nucleus and the interaction between nucleons. The basic building blocks of all nuclei are the protons and neutrons. The proton and neutron are regarded as two different aspects of the nucleon with different isospin. The study of the nucleus has taught us many new things about nature; the interaction between two nucleons is one of the central questions in nuclear physics. In this century, physicists have devoted a huge amount of experimentation and theoretical work to this problem.

The five most important properties of the nucleon-nucleon interaction are:

1. Nuclear force is of short range. That the range of nuclear force is shorter than interatomic distance we can conclude from the fact that at the molecular level we only need to consider the electromagnetic interaction. When going from the $A = 4$ nucleus, helium, upwards to higher-A nuclei, we realize that the binding energy per nucleon remains approximately constant. The density remains roughly the same, the radius of heavy nuclei being proportional to $A^{\frac{1}{3}}$. If the nuclear force was of long range, the potential energy per nucleon would increase with A and so would the density. On the other hand, the binding energy per nucleon grows with A for light nuclei($A \leq 4$).

2. The nuclear force is attractive in its intermediate range. The proof for the attractive character of the nuclear force is provided by the fact of nuclear binding.

3. The nuclear force has a repulsive core. Such an assumption could help explain the saturation properties of nuclear forces and the constant nuclear density.

4. There is a tensor force. The important piece of evidence for this fact are several properties of deuteron; the quadrupole moment, the magnetic moment(which requires a $D-$state contribution), and the asymptotic D/S ratio of the scattered wave. Further evidence is provided by the nonvanishing mixing parameters, ε_j, as obtained in a phase shift analysis of nucleon-nucleon scattering.

5. There is a spin-orbit force. A first indication for this fact was observed in the spectra of nuclei. Clear evidence came from the first reliable phase shift analysis at high energy. The triplet P wave resulting from the analysis can only be explained by assuming a strong spin-orbit force.

As we deal with the spin dependence of the nuclear force, we should also mention that there is a spin-spin force, which is not as important as the last two forces described above. Furthermore, the nuclear force is isospin dependent and is charge independent.

In order to learn detailed features of the nucleon-nucleon potential, we must study nucleon-nucleon scattering, and obtain the phase shift of the nucleon-nucleon interaction. A very important feature of nucleon-nucleon interaction is the reaction cross section. The cross section is closely related to the phase shift, so to study the phase shift of the nucleon-nucleon interaction is essential. The natural starting point for a study of the nucleon-nucleon interaction is the two nucleon system. The combination of the proton and neutron is a deuteron. This system possesses precisely one bound state. The properties of this bound state tell us a fair amount about the nuclear forces.

The first experimental data bearing directly on the nucleon-nucleon interaction came from low-energy scattering done in the 1930's. The data had little to say about the nature of the nuclear force. Almost any potential well with two adjustable parameters could be made to fit all low energy proton-proton(pp) data and a similar situation held in the neutron-proton(np) case.

In 1939, Rabi and co-workers[1] showed that the deuteron has an electric quadruple moment. This experimental result guaranteed that nuclei would be much more complicated structures than atoms for it implies that part of the NN interaction has a tensor character. In the later 1940's effective range theory explained the cause of the low energy " two-parameters-only" trouble mentioned above. Due to the short range of the nuclear force, scattering below 5 MeV takes place with the nucleons almost entirely in an S state and due to the strength of the force, the energy dependence of the S wave phase shift is adequately described by only two parameters.

Whereas at low energies the scattering process is very simple, at higher energies it is very complicated. For example, the pp scattering matrix contains the known Coulomb amplitude plus five unknown, linearly independent, complex functions of the energy and scattering angle. The effort involved in phase shift analyses of two nucleon interaction is massive, and we do not attempt to fit the data above about 350 MeV. At higher energies, pion production(within 280 MeV threshold) becomes important and both the two-nucleon Schrödinger equation and the two body Dirac equations are inadequate. We limit our fitting to elastic scattering below 350 MeV.

Because of progress of technology , the method to measure the more sophisticated spin observables, became feasible around 1957. Thus, in a very extensive set of experiments, Chamberlain and co-workers[2] made measurements on proton-proton scattering at Berkley. For the first time, a complete set of observables was measured. The phase shift analysis based on essentially these data performed by Stapp et al[3] has been of great importance to subsequent developments of phenomenological potentials.

The basic aim of a potential description of the two-nucleon interaction is twofold. One is to provide a summary of the data for comparison with potential-like results from theory. The other aim of a phenomenological potential is to serve as an input for nuclear structure calculations.

There have been several different approaches to the nucleon-nucleon scattering problem. Earlier approaches based on pure Feynman diagrams and pion exchange were not successful because of the large pion-nucleon coupling constant. The second set of approaches used phenomenological potentials. The phenomenological types of potentials have been improved over the past several decades. Later ambitious attempts have been much more successful. Examples of the hard-core type are those constructed by Hamada and Johnston[4] and by the Yale group[5]. These models employ a one-pion tail and reproduce the deuteron properties accurately. In the mid 1960s, Reid[6] developed hard and soft core potentials. His potentials became the most applied potentials in nuclear structure physics in the 1970s. Although Reid's phenomenological hard core and soft core potentials are widely accepted, his potentials lack physical content.

There are a lot of phenomenological potentials. Some of the most important are Reid's soft core potentials. He obtained a very good fit to the experimental data for states of singlet ${}^1S_0$, ${}^1P_1$, ${}^1D_2$, and triplet ${}^3P_0$, ${}^3P_1$, ${}^3S_1$, ${}^3D_1$, and he also included pp scattering for some states. He used the nonrelativistic Schrödinger equation, and used different potential forms for different angular momentum states. He assumed the potential in the asymptotic region was as a Yukawa form due to the pion exchange. Although his results are good, his potentials have different forms for different angular momentum states. Most terms in his potentials do not have a clear physical meaning. Because he used the nonrelativistic Schrödinger equation, when the energy is high, his potentials become inadequate.

The third category[7,12,40,41,42,43] of approaches to nucleon nucleon scattering problem combines Feynman diagrams with the relativistic wave equation. They have been very successful. Franz Gross[7] studied the nucleon-nucleon interaction relativistically, and he obtained semi-phenomenological potentials for singlet $^1S_0$ and triplet $^3P_0$, $^3P_1$, $^3S_1$, $^3D_1$ states, etc. Although he used relativistic wave functions, his equations can be reduced to Schrödinger-like equations only at the nonrelativistic limit, unlike the two body Dirac equations which can be reduced to Schrödinger-like equations without any limitation. His potentials[7] are nonlocal, velocity dependent and not symmetric under the exchange of two particles. He obtained his potentials by fitting to Reid's potentials rather than by fitting the experimental phase shift data. However, Reid obtained his soft core potentials by fitting the experimental phase shift data nonrelativistically. Gross obtained a good fit only for the singlet $^1S_0$ state. For all the other triplet $^3P_0$, $^3P_1$, $^3S_1$, $^3D_1$ states, Gross's potentials are just reasonably good[7].

In this dissertation, I obtain a semi-phenomenological relativistic potential model for nucleon-nucleon interactions by using two body Dirac equations and Yukawa's theory. One can derive the two-body Dirac equations for all Lorentz invariant interactions acting together or in any combinations. We also reduce the two-body Dirac equations to coupled Schrödinger-like equations, which unlike all the other relativistic semi-phenomenological approaches our potentials are local. By doing this, we can take advantage of earlier work done by the other people on the nonrelativistic Schrödinger equation in this way. Our potentials for different angular momentum states are constructed from the combinations of several different meson exchanges. Thus our potentials are semi-phenomenological potentials. Unlike Reid's potentials, every terms in our potential has clear physical meaning. Furthermore, our potentials, as well as the whole equation, are local. So it is possible that some different angular momentum states may have the exact same potentials, for example, singlet ${}^1S_0$ and ${}^1D_2$ states. It is the aim of this dissertation to see if our potentials are adequate to describe the elastic nucleon-nucleon interactions from low energy to high energy when using them together with two body Dirac equations of constraint dynamics[21,23,31].

Although many people work on this subject, all the other people's approaches have not been tested nonperturbatively in both QED and QCD. The two body Dirac equation approach yields manifestly covariant coupled equations for all the covariant interaction acting simultaneously. Unlike some of the other approaches, the relativistic spin corrections need not be treated only perturbatively. This mean that we can use nonperturbative methods(numerical methods) to solve the two body Dirac equations. This is a very important advantage of the two body Dirac equations and this approach has been successfully tested in QED and QCD. This gives us confidence that they are appropriate relativistic equations for phase shift analysis of nucleon-nucleon scattering. Since this approach is very useful in numerical calculation of atomic physics and particle physics, we propose to test this method in nuclear physics in the phase shift analysis nucleon-nucleon scattering.

To construct the potential of nucleon nucleon interaction is an old problem. A quote from an article entitled " What Holds the Nuclei Together" by Hans A. Bethe published in Scientific American in 1953[8] characterizes and summarizes the work done on nucleon-nucleon interaction:

> In the past quarter century physicists have devoted a huge amount of experimentation and mental labor to this problem, probably more man-hours than have been given to any other scientific questions in the history of mankind.

Although decades have elapsed, many physicists still work on this rocky subject, in which the efforts were larger by about several orders of magnitude than before.

In chapter 2, I discuss Yukawa's theory which is the basic idea of our meson exchange model, phenomenological potentials, semi-phenomenological potentials and the works done by other people in the area of nucleon-nucleon scattering phase shift analyses. All these works are closely related to my dissertation. In chapter 3, I introduce the two body Dirac equations of constraint dynamics. I mainly concentrate in the chapter on the deriving work I have done, which includes reducing the two-body Dirac equations to coupled Schrödinger-like equations and eliminating the first derivative terms which are inherent in the two body Dirac equations. This chapter is a crucial part of my dissertation. The detailed procedures of my derivations are presented in the Appendix B and Appendix C. In chapter 4, I discuss the phase shift methods used in our numerical calculations, which include phase shift equations for uncoupled and coupled states and the phase shift equations with Coulomb potential. In chapter 5, I present the models used in our calculations, including the expressions for the scalar, vector and pseudoscalar interactions, and the way they enter into our two body Dirac equations, mesons used in our fitting and the modification of our phase shift equations we found necessary. In chapter 6, I present the results we have achieved. Chapter 7 are the summaries and conclusions of our work.

Chapter 2

Nucleon-Nucleon Potential

In this chapter, I discuss Yukawa's theory, which is the basic idea of our meson exchange model, phenomenological potentials, semi-phenomenological potentials and the works done by other people in area of nucleon-nucleon scattering phase shift analyses. All these works are closely related to my dissertation.

The fundamental view of the nuclear force is that it arises from the exchange of mesons. Although one can infer the forms of potentials based on this view, one can not obtain the precise form of the potential in a convincing way starting from this picture. It proves to be the case that one can construct semiphenomenological potentials that account for the properties of the nuclear forces reasonably well for scattering energies up to a few hundred MeV. It is our aim to determine if such a semiphenomenological description is adequate for the N-N potential with two body Dirac equations of constraint dynamics.

We first consider the nonrelativistic Schrödinger equation for the two-nucleon system,

$$\left[\frac{\overrightarrow{p}^2}{2\mu} + V(r)\right]\psi = E\psi(r), \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \tag{2.1}$$

For uncoupled states, we have $s = 0$, $j = l$ or $s = 1$, $j = l$, for example, for ${}^1S_0$, ${}^1P_1$, ${}^1D_2$, and ${}^3P_1$ states, this leads to the radial Schrödinger equation

$$[\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} - U(r)]u(r) = 0 \tag{2.2}$$

$$k^2 = \frac{2\mu E}{\hbar^2}, \qquad U(r) = \frac{2\mu V(r)}{\hbar^2} \tag{2.3}$$

For coupled states, $s = 1$, $j = l \pm 1$. In general, the potentials consist of a central part $V_C(r)$, a tensor part $V_T(r)$, a spin-orbit part $V_{LS}(r)$ and a spin-spin part $V_{SS}(r)$

$$V(r) = V_C(r) + V_T(r)S_{12} + V_{LS}L \cdot S + V_{SS}\sigma_1 \cdot \sigma_2 \tag{2.4}$$

$$S_{12} = 3(\sigma_1 \cdot \hat{r})(\sigma_2 \cdot \hat{r}) - \sigma_1 \cdot \sigma_2 \tag{2.5}$$

The S_{12} operator is known as the tensor operator, since in configuration space it is a second order tensor rather than a scalar as was true for the central potential $V_c(r)$. Due to its tensor character in configuration space, it will admix states of different orbital angular momentum, in particular, for $l = j - 1$ and $l = j + 1$. Furthermore, in order to have an admixture of the two states with $l = j - 1$ and $l = j + 1$ we must clearly have spin 1 states(triplet states) and not singlet states. This means the tensor force is operative between triplet states only, for example, $^3S_1$ and $^3D_1$ states. This leads to the coupled radial Schrödinger equations

$$\begin{aligned} & [\frac{d^2}{dr^2} + k^2 - \frac{j(j-1)}{r^2} - U_C(r) - U_{ss}(r) + \frac{2(j-1)U_T(r)}{(2j+1)} - (j-1)U_{LS}(r)]u_-(r) \\ = \quad & \{\frac{6[j(j+1)]^{\frac{1}{2}}}{(2j+1)}\}U_T u_+(r) \end{aligned} \tag{2.6}$$

$$\begin{aligned} & [\frac{d^2}{dr^2} + k^2 - \frac{(j+2)(j+1)}{r^2} - U_C(r) - U_{ss}(r) + \frac{2(j+2)U_T(r)}{(2j+1)} + (j+2)U_{LS}(r)]u_+(r) \\ = \quad & \{\frac{6[j(j+1)]^{\frac{1}{2}}}{(2j+1)}\}U_T u_-(r) \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} k^2 &= \frac{2\mu E}{\hbar^2}, \\ U_C(r) &= \frac{2\mu V_C(r)}{\hbar^2}, \\ U_T(r) &= \frac{2\mu V_T(r)}{\hbar^2}, \\ U_{LS}(r) &= \frac{2\mu V_{LS}(r)}{\hbar^2}, \\ U_{SS}(r) &= \frac{2\mu V_{SS}(r)}{\hbar^2}, \end{aligned} \tag{2.8}$$

u_- is the wave function for $l = j - 1$, u_+ is the wave function for $l = j + 1$ channel and

j is the total angular momentum.

These radial Schrödinger equation are coupled, and the potential is in matrix form. For the states ${}^3P_0$ and ${}^3P_1$, these coupled equations become an uncoupled Schrödinger equation.

Even though the above equations are nonrelativistic Schrödinger equations, we shall see in the later section that the central part $V_C(r)$, tensor part $V_T(r)$, spin-orbit part $V_{LS}(r)$ and spin-spin part $V_{SS}(r)$ come out automatically after we reduce the two body Dirac equations to Schrödinger like equations. This is the primary value of the relativistic approach.

2.1 The One-Pion Exchange Potential (OPEP)

In 1935, Yukawa proposed that the force which acts between two nucleons may be produced by the exchange of mesons. This theory is an extension of ideas that arise in quantum electrodynamics and that describes the force acting between charged particles in terms of the exchange of photons.

When one meson is transferred between two nucleons. The intermediate state in this process contains a virtual meson connecting the initial and final states. This situation is of course permitted in quantum mechanics because of the uncertainty principle. However, it can only exist for a time interval compatible with the energy uncertainty, that is, the lighter the meson the longer the ranger of the force component which it can carry. The lightest meson is the pion , and it will mediate the force component of longest range. We may extend the Yukawa theory beyond the OPEP. We may do so by considering the exchange of several pions at once, but it is more efficient to use a wave equation than to use more complicated Feynman diagrams with just pion exchanges. Because the coupling constant of nucleon-nucleon interaction mediated by pions is large, it may be questionable whether a perturbation expansion make any sense. On the other hand, using the single pion exchange potential in either a nonrelativistic or relativistic wave equation efficiently iterates the single pion exchange to all orders. This can be explained by the following:[9]

For the higher order Born approximation, we define the transition operator T such that

$$V \mid \psi^{(+)}\rangle = T \mid \phi\rangle, \tag{2.9}$$

where $\mid \phi\rangle$ is the solution to the free-particle Schrödinger equation

$$H_0 \mid \phi\rangle = E \mid \phi\rangle, \tag{2.10}$$

and $\mid \psi^{(+)}\rangle$ is a plane wave in propagation direction k plus an outgoing spherical wave of scattering amplitude $f(k', k)$:

$$\langle x \mid \psi^{(+)}\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}}[e^{ik\cdot x} + \frac{e^{ikr}}{r} f(k', k)]. \tag{2.11}$$

This leads to the Lippmann-Schwinger equation

$$T \mid \phi\rangle = V \mid \phi\rangle + VGT \mid \phi\rangle \tag{2.12}$$

where

$$G = \frac{1}{E - H_0 + i\varepsilon}. \tag{2.13}$$

This hold for $\mid \phi\rangle$ to be any plane-wave state; furthermore, we know $\mid \phi\rangle$ are complete. so we have

$$T = V + VGT. \tag{2.14}$$

We can obtain an iterative solution for T as follows:

$$T = V + VGV + VGVGV + \cdots \tag{2.15}$$

the scattering amplitude $f(k', k)$ can be written as

$$f(k', k) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\langle k' \mid T \mid k\rangle \tag{2.16}$$

Thus to determine $f(k', k)$, it is sufficient to know the transition operator T. We can expand $f(k', k)$ as follows:

$$f(k', k) = \sum_{n=1}^{\infty} f^{(n)}(k', k) \tag{2.17}$$

where n is the number of times the V operator enters. we have

$$f^{(1)}(k',k) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\langle k' \mid V \mid k\rangle, \tag{2.18}$$

$$f^{(2)}(k',k) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\langle k' \mid VGV \mid k\rangle, \tag{2.19}$$

$$\ldots$$

$$etc$$

Solving the wave equation nonperturbatively for phase shift is equivalent to finding T, and that in turn is equivalent to iterating the Lippmann-Schwinger equation to all orders. This has the effect of including multiple pion exchanges in an approximate way, provided the scattering potential V is adequate.

We may also supplement the OPEP in the wave equation by considering the exchange of single bosons other than the pion; the corresponding potentials form the class of one-boson exchange potentials (OBEP). They may be constructed from the observed properties of the various strongly interacting bosons (mesons); Such as those listed in Table 2.1.

In some case it may be necessary to postulate the existence and properties of fictitious mesons. The need to introduce such mesons in the OBEP arises because they simulate effects that correspond to aspects of the exchange of several mesons not accounted for by the iteration of the Lippmann-Schwinger equation with the given potentials. Such

Table 2.1: Data On Mesons(T=isospin, G=G-parity, J=spin, π =parity)

Particles	Mass(MeV)	T^G	J^π	Width(MeV)
$\pi^\pm$	139.57018±0.00035	1^-	0^-	—
π^0	134.9766 ± 0.0006	1^-	0^-	—
η	547.3 ± 0.12	0^+	0^-	$(1.18 \pm 0.11) \times 10^{-3}$
ρ	769.3 ± 0.8	1^+	1^-	150.2 ± 0.8
ω	782.57 ± 0.12	0^-	1^-	8.44 ± 0.09
$\eta^{'}$	957.78 ± 0.14	0^+	0^-	0.202 ± 0.016
ϕ	1019.417± 0.014	0^-	1^-	4.458 ± 0.032
f_0	980 ± 10	0^+	0^+	40 to 100
a_0	984.8 ± 1.4	1^-	0^+	50 to 100
σ	500–700	0^+	0^+	600 to 1000

Source: Review of Particle Physics, D. E. Groom et al., The European Physical Journal C15 (2000) 1.

is the case of the very broad sigma meson(σ).

A derivation of the complete two nucleon potential from the meson theory would help greatly to solve the problems in the N-N interaction, but this does not seem possible. Therefore we have had to confine ourselves to phenomenological potentials or semi-phenomenological potentials based on the concept of meson theory in the context of the Schrödinger equation or two-body Dirac equations.

From the practical point of view, the meson-exchange theory is a useful starting point for studying the nucleon-nucleon interaction. In Yukawa theory, the interaction between two nucleons is mediated by the exchange of various mesons. Although it is extremely difficulty to make a quantitative connection with the underlying quark structure of the hadrons, the theory makes it possible to relate the nuclear interaction with various other hadronic processes, such as the strength of meson-nucleon interactions.

The Yukawa theory provides us with a reasonable form of the radial dependence for a nuclear potential. Such a form may be used as the starting point for constructing the phenomenological potentials and semi-phenomenological potentials.

The Yukawa potential form is

$$\phi(r) = \frac{g}{4\pi r} e^{-\frac{mc}{\hbar} r}, \tag{2.20}$$

here m is the exchanged meson mass. $\phi(r)$ reduces to Coulomb potential on letting $m = 0$, and $g = 4\pi q$. In this case, the exchanged particle is a photon, which is massless. On the other hand, if the field quantum has finite mass, we find the strength of potential drops by a factor $1/e$ at a distance $r_0 = \hbar/mc$. The quantity r_0 may be taken as a measure of the range of the force mediated by a meson of mass m. For pions($m = 140$ MeV/c^2), the value of r_0 is around 1.4 fm. For sigma meson($m = 500$ MeV/c^2), the value of r_0 is around 0.4 fm. For ρ meson ($m = 770$ MeV/c^2), the value of r_0 is 0.254 fm. For ω meson ($m = 783$ MeV/c^2), the value of r_0 is around 0.25 fm. We will see later that the exchange of a single pion gives a good representation of the long-range part of the nuclear potential.

The present view is that the nuclear force may be divided into three parts. The longest part($r > 2$ fm) is dominated by one pion exchange. If the exchange of single meson is important, there is no reason to exclude exchange of mesons heavier than pions. The range of force associated with these more massive bosons is shorter, the intermediate range part of nuclear force (1 $fm > r > 2$ fm) comes mainly from the simultaneous exchange of two pions and heavier mesons. The shorter range ($r \leq 1\ fm$) in the interaction of two nucleon is made of heavy mesons exchange. The pion mesons mediate the longest range of nucleon-nucleon interaction, the σ and η mesons mediate the intermediate range of the nucleon-nucleon interaction, the ρ, ω, ϕ, a_0, f_0 and η' mesons, etc, mediate the short range of the nucleon-nucleon interaction.

2.2 Phenomenological Potentials

Potentials based on the Yukawa theory can achieve some fair success. A phenomenological potential is one which displays the two-nucleon Yukawa interaction behavior in the asymptotic region but in other regions with a behavior having no clear physical connection to meson exchanges.

There are a lot of phenomenological potentials. Three of the most used phenomenological hard core or soft core potentials in nuclear structure theory are Hamada and Johnston, Yale and Reid's phenomenological potentials[6,10,11]. Hard core potential mean wave functions vanish at nonzero radii, that is the potential becomes infinity at nonzero radii. Soft core potential mean wave functions do not vanish at nonzero radii; the potential is finite for finite radii. The longest range part of these phenomenological potentials is the OPEP and has the following form:

$$V^{OPEP} = (g^2/12)m_\pi c^2(m_\pi/M)^2\tau_1 \cdot \tau_2 \left[\sigma_1 \cdot \sigma_2 + S_{12}(1 + \frac{3}{x} + \frac{3}{x^2})\right] e^{-x}/x, \qquad (2.21)$$

where m_π is the pion mass, M is the nucleon mass, $< \tau \cdot \tau_2 >$ is 1 or -3 for isotopic spin 1 or 0 and $x = \mu r$ with $\mu = m_\pi c/\hbar$. This constitutes most of the N-N interaction at distances greater than about 2 or 3 Fermis. All models contain this part.

The Hamada and Johnston potentials are[10]

$$V(r) = V_C(r) + V_T(r)S_{12} + V_{LS}L \cdot S + V_{LL}L_{12}, \tag{2.22}$$

where

$$L_{12} = (\sigma_1 \cdot \sigma_2)L^2 - \frac{1}{2}[(\sigma_1 \cdot L)(\sigma_2 \cdot L) + (\sigma_2 \cdot L)(\sigma_1 \cdot L)] = [\delta_{LJ} + \sigma_1 \cdot \sigma_2]L^2 - (L \cdot S)^2 \tag{2.23}$$

and

$$V_C = 0.08 \cdot (1/3)m_\pi c^2(\tau_1 \cdot \tau_2)(\sigma_1 \cdot \sigma_2)(1 + a_C\frac{e^{-x}}{x} + b_C\frac{e^{-2x}}{x^2}) \tag{2.24}$$

$$V_T = 0.08 \cdot (1/3)m_\pi c^2(\tau_1 \cdot \tau_2)(1 + \frac{3}{x} + \frac{3}{x^2})(1 + a_T\frac{e^{-x}}{x} + b_T\frac{e^{-2x}}{x^2})\frac{e^{-x}}{x} \tag{2.25}$$

$$V_{LS} = m_\pi c^2 G_{LS}(1 + b_{LS}\frac{e^{-x}}{x})\frac{e^{-2x}}{x^2} \tag{2.26}$$

$$V_{LL} = m_\pi c^2\frac{1}{x^2}(1 + \frac{3}{x} + \frac{3}{x^2})(1 + a_{LL}\frac{e^{-x}}{x} + b_{LL}\frac{e^{-2x}}{x^2})\frac{e^{-x}}{x} \tag{2.27}$$

and $x = \mu r$ with $\mu = m_\pi c/\hbar$. In the Hamada and Johnston potentials, the hard core is accounted for by setting all the potentials $V(r)$ equal to infinity for $x \leq 0.343$. The choice of the constant obtained by Hamada and Johnston after fitting the experimental phase shift data for the two nucleon system.

The Yale potentials[11] are given by

$$V(r) = V^{OPEP} + V_C(r) + V_T(r)S_{12} + V_{LS}L \cdot S + V_q[(L \cdot S)^2 + L \cdot S - L^2], \qquad (2.28)$$

where

$$V_\beta = \sum_n a_n^{(\beta)} x^{-n} e^{-2x}, \qquad \beta = C, T, LS, q. \qquad (2.29)$$

In the V^{OPEP}, the pion-nucleon coupling constant g^2 is 14×0.94 for singlet even states, and is 14×1 otherwise. Also the mass of the pion is replaced with $m_\pi = m_{\pi^0}$, for single even states and triplet odd states, $m_\pi = \frac{2}{3}m_{\pi^\pm} + \frac{1}{3}m_{\pi^0}$,for single odd states and triplet even states.

Reid's potential has the same OPEP behavior in the asymptotic region. He obtained two set of potentials, the hard core potentials and the soft core potentials. His choice for g^2 is 14. At intermediate distance, the potential was represented by sums of the convenient Yukawa form, $\exp(-nx)/x$, where n is an integer. No attempt was made to make $n \cdot m_\pi$ correspond to the mass of a known particle. He expressed the short range repulsion by means of soft (Yukawa) cores. Their Yukawa cores are soft in the sense that wave functions do not vanish inside them at nonzero radii. Some of his soft core potentials are listed below[6]

Soft core potentials $T = 1$, $h = 10.463$, and $x = \mu r$ with $\mu = m_\pi c/\hbar$:

$$V(^1S_0) = -h\frac{e^{-x}}{x} - 1650.6\frac{e^{-4x}}{x} + 6484.2\frac{e^{-7x}}{x}, \tag{2.30}$$

$$V(^1D_2) = -h\frac{e^{-x}}{x} - 12.322\frac{e^{-2x}}{x} - 1112.6\frac{e^{-4x}}{x} + 6484.2\frac{e^{-7x}}{x}, \tag{2.31}$$

$$V(^3P_0) = -h[(1+\frac{4}{x}+\frac{4}{x^2})e^{-x}-(\frac{16}{x}+\frac{4}{x^2})e^{-4x}]/x+27.133\frac{e^{-2x}}{x}-790.74\frac{e^{-4x}}{x}+20662\frac{e^{-7x}}{x}, \tag{2.32}$$

$$V(^3P_1) = -h[(1+\frac{2}{x}+\frac{2}{x^2})e^{-x} - (\frac{8}{x}+\frac{2}{x^2})e^{-4x}]/x - 135.25\frac{e^{-2x}}{x} + 472.81\frac{e^{-3x}}{x}. \tag{2.33}$$

Two alternate potentials are

$$V(^1S_0, alternate) = -h\frac{e^{-x}}{x} + 105.32\frac{e^{-3x}}{x} - 2401.9\frac{e^{-4x}}{x} + 5598.2\frac{e^{-6x}}{x}, \tag{2.34}$$

$$V(^1D_2, alternate) = -h\frac{e^{-x}}{x} - 318.64\frac{e^{-3x}}{x} + 526.27\frac{e^{-5x}}{x}, \tag{2.35}$$

Soft core potentials $T=0$,

$$V(^1P_1) = 3h\frac{e^{-x}}{x} - 634.39\frac{e^{-2x}}{x} + 2163.4\frac{e^{-3x}}{x}, \tag{2.36}$$

$$V(^1D_2) = -3h[(1+\frac{2}{x}+\frac{2}{x^2})e^{-x} - (\frac{8}{x}+\frac{2}{x^2})e^{-4x}]/x - 220.12\frac{e^{-2x}}{x} + 871\frac{e^{-3x}}{x}, \tag{2.37}$$

$$V(^3S_1 -^3 D_1) = V_c(r) + V_T(r)S_{12} + V_{LS}L\cdot S, \tag{2.38}$$

where

$$V_C = -h\frac{e^{-x}}{x} + 105.468\frac{e^{-2x}}{x} - 3187.8\frac{e^{-4x}}{x} + 9924.3\frac{e^{-6x}}{x}, \tag{2.39}$$

$$V_T = -h[(1+\frac{3}{x}+\frac{3}{x^2})e^{-x} - (\frac{12}{x}+\frac{3}{x^2})e^{-4x}]/x + 351.77\frac{e^{-4x}}{x} - 1673.5\frac{e^{-7x}}{x}, \tag{2.40}$$

$$V_{LS} = 708.91\frac{e^{-4x}}{x} - 2713.1\frac{e^{-6x}}{x}. \tag{2.41}$$

Two alternate potentials are

$$V(^3S_1 -^3 D_1, alternate) = V_c(r) + V_T(r)S_{12} + V_{LS}L \cdot S, \tag{2.42}$$

where

$$V_C = -h\frac{e^{-x}}{x} + 102.012\frac{e^{-2x}}{x} - 2915\frac{e^{-4x}}{x} + 7800\frac{e^{-6x}}{x}, \tag{2.43}$$

$$V_T = -h[(1 + \frac{3}{x} + \frac{3}{x^2})e^{-x} - (\frac{12}{x} + \frac{3}{x^2})e^{-4x}]/x + 163\frac{e^{-4x}}{x}, \tag{2.44}$$

$$V_{LS} = 251.57\frac{e^{-4x}}{x}, \tag{2.45}$$

$$V(^1P_1, alternate) = 3h\frac{e^{-x}}{x} - 240\frac{e^{-2x}}{x} + 17000\frac{e^{-6x}}{x}. \tag{2.46}$$

Reid's potentials are local, but a different potential is used for each state of distinct isotopic spin, total spin, and total angular momentum(similar ad hoc assumptions are made by Hamada and Johnson and Yale group). He also obtained several alternate potentials for some states. Potential $V(^1S_0)$ and $V(^1S_0, Alternate)$ have the different form, but they are fitted from the same experimental data. Likewise, for $V(^3S_1 -^3 D_1)$ and $V(^3S_1 -^3 D_1, alternate)$. Alternate potentials are a feature of the phenomenological potential. Because of the lack of a clear physical meaning for most of the terms in the phenomenological potential, we may obtain another set of alternate potentials by changing the potential forms. If every term in the potentials have clear physical meaning and origin, we can not obtain the alternate potentials because we do not have the freedom to change the form of the potentials although coupling constant may vary.

In Reid's nonrelativistic phenomenological approach, the central part $V_C(r)$, tensor part $V_T(r)$ and spin-orbit part $V_{LS}(r)$ are fitted separately for each angular momentum. For example, for the triplet $^3S_1$ and $^3D_1$ state soft core potential, Reid's central part $V_C(r)$, tensor part $V_T(r)$ and spin-orbit part $V_{LS}(r)$ are given by

$$V_C = -h\frac{e^{-x}}{x} + 105.468\frac{e^{-2x}}{x} - 3187.8\frac{e^{-4x}}{x} + 9924.3\frac{e^{-6x}}{x}, \tag{2.47}$$

$$V_T = -h[(1 + \frac{3}{x} + \frac{3}{x^2})e^{-x} - (\frac{12}{x} + \frac{3}{x^2})e^{-4x}]/x + 351.77\frac{e^{-4x}}{x} - 1673.5\frac{e^{-7x}}{x}, \tag{2.48}$$

and

$$V_{LS} = 708.91\frac{e^{-4x}}{x} - 2713.1\frac{e^{-6x}}{x}. \tag{2.49}$$

Every term in the central part $V_c(r)$, tensor part $V_T(r)$ and spin-orbit part $V_{LS}(r)$ are independent from each other. This means that Reid can put any terms he wants and remove any terms he does not want. This makes his fitting much easier. In the following chapters, I show that every terms in the central part $V_c(r)$, tensor part $V_T(r)$ and spin-orbit part $V_{LS}(r)$ of our semi-phenomenological potentials are correlated by the physics of the two body Dirac equations, and hence we do not have the freedom that Reid had when we fit the experimental data.

2.3 Semi-Phenomenological Potentials

In this approach the behavior of the two-nucleon interaction semi-phenomenological potential in the asymptotic region is a Yukawa potential due to pion exchange and in other regions is given by fixed potential forms due to exchanges of several different types of mesons. By semi-phenomenological potential, we mean that we just fit the coupling constants not the actual form of the potentials as Reid did in his phenomenological potential fitting.

Franz Gross used a relativistic , three dimensional wave equation which restricts one of the two particles to its mass shell and applied it to the study of nucleon-nucleon scattering phase shifts. He examined a simple model in which the nuclear force is represented by the exchange of four mesons; the π, ρ, ω, σ. In order to study the dynamics of his equation without solving for the phase shifts, he took the nonrelativistic limit and obtained a Schrödinger equation with an effective potential, which could be compared with Reid's phenomenological potentials. This limiting process is very well defined, but its accuracy is doubtful, particularly at short distance. He fixed g_π^2 to agree with Reid's $g_\pi^2/4\pi = 14.0$, so that Gross's long range part of the OPEP is identical to Reid's. By adjusting some of the coupling constants and taking the nonrelativistic limit he obtained a fit to the phenomenological soft core potentials previously obtained by Reid.

In momentum space, Gross's quasipotential equation is

$$(\Gamma C)_{\mu\nu}(\widehat{p}) = -\int \frac{d^3k}{(2\pi)^3} V_{\mu\mu',\nu\nu'}(\widehat{p},\widehat{k},W) G_{\mu'\mu'',\nu'\nu''}(\widehat{k},W)(\Gamma C)_{\mu''\nu''}(\widehat{k})], \tag{2.50}$$

where μ and ν are spinor indices, $P = (W, 0)$ is the total energy-momentum 4 vector, p and k are relative 4 momenta, V is the interaction kernel with particle 1 on the mass shell, C is the charge conjugation matrix, $\Gamma_{\mu\nu}$ is the covariant two nucleon vertex function. G is two body Green's function

$$G_{\mu'\mu'',\nu'\nu''}(\widehat{k},W) = \frac{[M + \gamma\cdot(\frac{P}{2}+\widehat{k})]_{\mu'\mu''}[M + \gamma\cdot(\frac{P}{2}-\widehat{k})]_{\nu'\nu''}}{2E_k W(2E_k - W)}, \tag{2.51}$$

where

$$\widehat{k} = (\widehat{k}_0, \widetilde{k}); \qquad \widehat{p} = (\widehat{p}_0, \widetilde{p}); \tag{2.52}$$

$$\widehat{k}_0 = E_k - \frac{W}{2} \qquad \widehat{p}_0 = E_p - \frac{W}{2} \tag{2.53}$$

$$E_k = (M^2 + \widetilde{k}^2)^{\frac{1}{2}} \tag{2.54}$$

Gross's explicit form of relativistic wave equations and potentials can be written as

$$(2E_p - W)\psi^+_{rs}(\widetilde{p}) = -\int \frac{d^3k}{(2\pi)^3}[V^{++}_{rr',ss'}(\widetilde{p},\widetilde{k},W)\psi^+_{r's'}(\widetilde{k}) + V^{+-}_{rr',ss'}(\widetilde{p},\widetilde{k},W)\psi^-_{r's'}(\widetilde{k})], \tag{2.55}$$

$$-W\psi^{-}_{rs}(\widetilde{p}) = -\int \frac{d^3k}{(2\pi)^3}[V^{-+}_{rr',ss'}(\widetilde{p},\widetilde{k},W)\psi^{+}_{r's'}(\widetilde{k}) + V^{--}_{rr',ss'}(\widetilde{p},\widetilde{k},W)\psi^{-}_{r's'}(\widetilde{k})], \qquad (2.56)$$

where

$$\psi^{+}_{rs}(\widetilde{p}) = \frac{M}{\sqrt{2W}}\frac{u^{-(r)}_{\mu}(\widetilde{p})u^{-(s)}_{\nu}(-\widetilde{p})(\Gamma C)_{\mu\nu}(\widehat{p})}{E_p(2E_p - W)}, \qquad (2.57)$$

$$\psi^{-}_{rs}(\widetilde{p}) = -\frac{M}{\sqrt{2W}}\frac{u^{-(r)}_{\mu}(\widetilde{p})u^{-(s)}_{\nu}(\widetilde{p})(\Gamma C)_{\mu\nu}(\widehat{p})}{E_p W}. \qquad (2.58)$$

Hence his potentials are defined as

$$V^{++}_{1,2}(\widetilde{p},\widetilde{k},W) = (\frac{M^2}{E_p E_k})\overline{u}^{(r)}_{\mu}(\widetilde{p})\overline{u}^{-(s)}_{\nu}(-\widetilde{p})V_{\mu\mu',\nu\nu'}(\widetilde{p},\widetilde{k},W)u^{(r')}_{\mu'}(\widetilde{k})u^{(s')}_{\nu'}(-\widetilde{k}), \qquad (2.59)$$

$$V^{+-}_{1,2}(\widetilde{p},\widetilde{k},W) = (\frac{M^2}{E_p E_k})\overline{u}^{(r)}_{\mu}(\widetilde{p})\overline{u}^{-(s)}_{\nu}(-\widetilde{p})V_{\mu\mu',\nu\nu'}(\widetilde{p},\widetilde{k},W)u^{(r')}_{\mu'}(\widetilde{k})v^{(s')}_{\nu'}(\widetilde{k}), \qquad (2.60)$$

$$V^{-+}_{1,2}(\widetilde{p},\widetilde{k},W) = (\frac{M^2}{E_p E_k})\overline{u}^{(r)}_{\mu}(\widetilde{k})\overline{v}^{-(s)}_{\nu}(\widetilde{p})V_{\mu\mu',\nu\nu'}(\widetilde{p},\widetilde{k},W)u^{(r')}_{\mu'}(\widetilde{k})u^{(s')}_{\nu'}(-\widetilde{k}), \qquad (2.61)$$

$$V_{1,2}^{--}(\widetilde{p},\widetilde{k},W)=(\frac{M^2}{E_pE_k})\overline{u}_{\mu}^{(r)}(\widetilde{p})\overline{v}_{\nu}^{-(s)}(\widetilde{p})V_{\mu\mu',\nu\nu'}(\widetilde{p},\widetilde{k},W)u_{\mu'}^{(r')}(\widetilde{k})v_{\nu'}^{(s')}(\widetilde{k}). \tag{2.62}$$

The exact details of Gross's relativistic and semi-phenomenological wave equations are not important to explain our points. We observe that like many other such equations, it is nonlocal[12] in momentum or coordinate space.

To make this clear, consider a simple two body relativistic Hamiltonian operator for equal mass particles

$$H=2\sqrt{m^2+p^2}+V(r). \tag{2.63}$$

If this is expressed in coordinate space, then the square root operator is nonlocal; if this is expressed in momentum space (as Gross did), then the potential energy is nonlocal(the nonlocality in Gross's equation is represented by an integral form of the potential energy).

To obtain the nonrelativistic limit of these wave equation and potentials, Gross takes the limit in which the external 3-momentum $\widetilde{p}$, internal 3-momentum $\widetilde{k}$, and $\epsilon=W-2M$, can all be regarded as small compared to M. The assumption that $\widetilde{k}$ is small compared to M requires that the above wave equation will be dominated by small values of $\widetilde{k}$, which in turn will be true only if the range of the force is large compared to M^{-1}. This assumption is not very good, but it gives some physical insight into the nuclear force. In the nonrelativistic limit, Gross's relativistic wave equation becomes a Schrödinger-like equation of the form

$$-(\frac{\nabla^2}{M} + \epsilon)\psi_T = -V\psi_T. \tag{2.64}$$

Gross's nonrelativistic potentials are in the form

$$V(r) = V_c(r) + V_{SS}\sigma_1 \cdot \sigma_2 + V_T(r)S_{12} + V_{LS}L \cdot S + V_{LD}L \cdot D, \tag{2.65}$$

where

$$V_C(r) = \frac{D}{D_T}U_C + V_C^Q, \tag{2.66}$$

$$V_{SS}(r) = \frac{D}{D_T}U_{SS} + V_{SS}^Q, \tag{2.67}$$

$$V_T(r) = \frac{D}{D_T}U_T + V_T^Q, \tag{2.68}$$

$$V_{LS}(r) = \frac{D}{D_T}U_{LS} + V_{LS}^Q, \tag{2.69}$$

$$V_{LD} = V_{LD}^Q, \tag{2.70}$$

where

$$L \cdot D = \frac{1}{2} L \cdot (\sigma_1 - \sigma_2), \tag{2.71}$$

$$U_C = -V_0^\sigma + V_0^\omega + (\tau_1 \cdot \tau_2) V_0^\rho, \tag{2.72}$$

$$U_{SS} = (\tau_1 \cdot \tau_2)[V_0^\pi + \frac{m_\rho^2}{6M^2}(1 + K_\rho)^2 V_0^\rho] + \frac{m_\omega^2}{6M^2}(1 + K_\omega)^2 V_0^\rho, \tag{2.73}$$

$$U_T = (\tau_1 \cdot \tau_2)[V_2^\pi - (1 + K_\rho)^2 V_2^\rho] - (1 + K_\omega)^2 V_2^\omega, \tag{2.74}$$

$$U_{LS} = -\frac{m_\pi}{Mx}\{V_1^\sigma + (\tau_1 \cdot \tau_2)(1.5 + 2K_\rho) V_1^\rho + (1.5 + 2K_\omega) V_1^\omega\}, \tag{2.75}$$

$$D_T = D + \frac{1}{2} v_l^2, \tag{2.76}$$

$$D = 1 - \frac{1}{2M}(V_0^\sigma + V_0^\omega + (\tau_1 \cdot \tau_2) V_0^\rho), \tag{2.77}$$

$$v_l = \frac{1}{M}\{V_0^\sigma + (\tau_1 \cdot \tau_2) V_0^\rho + V_0^\omega\}, \tag{2.78}$$

Let $x = m_\pi r$, then

$$V_0^\pi = \frac{g_\pi^2}{4\pi}\frac{m_\pi^3}{12M^2}\frac{e^{-x}}{x}, \tag{2.79}$$

$$V_1^\pi = \frac{g_\pi^2}{4\pi}\frac{m_\pi^2}{2M}(1+\frac{1}{x})\frac{e^{-x}}{x}, \tag{2.80}$$

$$V_2^\pi = \frac{g_\pi^2}{4\pi}\frac{m_\pi^3}{12M^2}(1+\frac{3}{x}+\frac{3}{x^2})\frac{e^{-x}}{x} \tag{2.81}$$

Let $\sigma = \frac{m_\sigma}{m_\pi}$, then

$$V_0^\sigma = \frac{g_\sigma^2}{4\pi}m_\pi\frac{e^{-\sigma x}}{x}, \tag{2.82}$$

$$V_1^\sigma = \frac{g_\sigma^2}{4\pi}\frac{m_\pi^2}{2M}(\sigma+\frac{1}{x})\frac{e^{-\sigma x}}{x}. \tag{2.83}$$

Let $\rho = \frac{m_\rho}{m_\pi}$, then

$$V_0^\rho = \frac{g_\rho^2}{4\pi}m_\pi\frac{e^{-\rho x}}{x}, \tag{2.84}$$

$$V_1^\rho = \frac{g_\rho^2}{4\pi}\frac{m_\pi^2}{M}(\rho+\frac{1}{x})\frac{e^{-\rho x}}{x}, \tag{2.85}$$

$$V_2^\rho = \frac{g_\rho^2}{4\pi}\frac{m_\pi^2}{12M}(\rho^2+\frac{3\rho}{x}+\frac{3}{x^2})\frac{e^{-\rho x}}{x}. \tag{2.86}$$

Let $\omega = \frac{m_\omega}{m_\pi}$, then

$$V_0^\omega = \frac{g_\omega^2}{4\pi} m_\pi \frac{e^{-\omega x}}{x}, \tag{2.87}$$

$$V_1^\omega = \frac{g_\omega^2}{4\pi} \frac{m_\pi^2}{M} (\omega + \frac{1}{x}) \frac{e^{-\omega x}}{x}, \tag{2.88}$$

$$V_2^\omega = \frac{g_\omega^2}{4\pi} \frac{m_\pi^2}{12M} (\omega^2 + \frac{3\omega}{x} + \frac{3}{x^2}) \frac{e^{-\omega x}}{x}. \tag{2.89}$$

The quadratic contributions to each potential are given below

$$8MD_T V_C^Q = 2v_+^2 + v_a^2 + v_b^2 - \frac{2v_l}{r}(2v_+ - v_a - v_b) - D[\frac{v_l}{D}(2v_+ - v_a - v_b)]',$$

$$+4M\epsilon v_l^2 - \frac{2v_l}{D_T}(v_l' - \frac{v_l D'}{2D})^2 + \frac{8v_l}{r}(v_l' - \frac{v_l D'}{2D}) + 4(v_l' - \frac{v_l D'}{D})^2 + 4v_l(v_l'' - \frac{v_l D''}{2D}), \tag{2.90}$$

$$8MD_T V_{SS}^Q = \frac{2}{3}v_+^2 + \frac{1}{3}v_a^2 - v_b^2 - \frac{2v_l}{r}(\frac{2}{3}v_+ - \frac{1}{3}v_a + v_b) - D[\frac{v_l}{D}(\frac{2}{3}v_+ - \frac{1}{3}v_a + v_b)]', \tag{2.91}$$

$$12MD_T V_T^Q = v_+^2 - v_a^2 + \frac{v_l}{r}(v_+ + v_a) - D[\frac{v_l}{D}(v_+ + v_a)]', \tag{2.92}$$

$$2MD_TV^Q_{LS} = -v_l(v_+ - v_a) + D[\frac{v_l^2}{D}]', \tag{2.93}$$

$$2MD_TV^Q_{LD} = v_l(v_+ - v_b) - D[\frac{v_l^2}{D}]', \tag{2.94}$$

where the v's are defined as

$$v_l = \frac{1}{M}[V^\sigma_0 + (\tau_1 \cdot \tau_2)V^\rho_0 + V^\omega_0], \tag{2.95}$$

$$v_+ = -V^\sigma_1 + \frac{1}{2}K_\omega V^\omega_1 + (\tau_1 \cdot \tau_2)[V^\pi_1 + \frac{1}{2}K_\rho V^\rho_1], \tag{2.96}$$

$$v_a = V^\sigma_1 - (1 + \frac{3}{2}K_\omega)V^\omega_1 + (\tau_1 \cdot \tau_2)[V^\pi_1 - (1 + \frac{3}{2}K_\rho)V^\rho_1], \tag{2.97}$$

$$v_b = V^\sigma_1 + (1 + \frac{1}{2}K_\omega)V^\omega_1 + (\tau_1 \cdot \tau_2)[V^\pi_1 + (1 + \frac{1}{2}K_\rho)V^\rho_1]. \tag{2.98}$$

Up to now, it is obvious that Gross's potentials are in a very complicated form. His relativistic wave equation can be reduced Schrödinger-like equation only in nonrelativistic limit. He made some assumptions and ignored a lot of hard handled nonlocal potential terms before he compared his potentials with Reid's phenomenological soft core potentials. In contrast to Gross's relativistic and semi-phenomenological wave equations, we find that the two body Dirac equations of constraint dynamics can be exactly reduced to local Schrödinger-like equations. This allows us to gain physical insight into the nucleon nucleon interactions without making any assumption that are questionable.

Chapter 3

Two Body Dirac Equations

In this chapter, I introduce the two body Dirac equations of constraint dynamics. The materials presented here(up to Eq.(3.173) are reviewing the previous work done by other people. My work start from Eq.(3.173)). I mainly concentrate on the deriving work I have done, which include reducing the two-body Dirac equations to a coupled Schrödinger-like equations and getting rid of the first derivative terms which results from the reduction of the two body Dirac equations. This chapter is a crucial part of my dissertation. The detailed procedures of my derivations are presented in Appendix B and Appendix C.

It is our aim to obtain a semi-phenomenological fit to the experimental phase shift data by incorporating the meson exchange model and the two body Dirac equations of constraint dynamics. I am studying nucleon-nucleon interaction by using two body Dirac equations. Dirac proposed single particle Dirac equation in 1928[13]; it has been successfully applied to describe the relativistic effect in hydrogen-like systems. This is the single particle Dirac equation.

$$[\alpha \cdot \mathbf{p} + \beta m + V(r)]\,\psi = E\psi \tag{3.1}$$

α and β are 4×4 matrix, their expressions are not unique, in the Pauli-Dirac representation, they are in the form

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{3.2}$$

where σ is the Pauli matrix and I is the unity matrix. $V(r) = A(r) + \beta S(r)$ is for interactions that transforms as a world scalar and time-component four vector.

Its successful application to hydrogen-like systems is based on the assumption that the proton mass is much larger than the electron mass, so we can consider the proton as static and its magnetic moment can be ignored compared to the electron magnetic moment(there would be no dipole-dipole electromagnetic interaction between the proton and electron). The single particle Dirac equation is not suitable to describe the system such as the mesons,(quarkonium), muonium, positronium and deuteron, because the particles may have equal or near equal mass, and there is a dipole-dipole electromagnetic interaction between the particles that is not negligible. However, two body Dirac equations can solve these problems.

Theories that put both particles on an equal footing in describing two interacting spin

$\frac{1}{2}$ particles have all built on the single particle Dirac equation. In 1929, G. Breit[14,15,16] introduced an extension of the single particle Dirac equation to a two-body equation. However, the Breit equations do not retain the manifest covariant form and his equations can not be treated non-perturbatively beyond the Coulomb term[15].

Several authors[17−28] have applied the approach of Dirac's Hamiltonian constraint dynamics to two body bound state wave equations and it appears to have been quite successful and shows additional promise. This approach provides a manifestly covariant detour around many of the problems that hamper the implementation and application of Breit's two body Dirac equations and it also can give us a local Schrödinger-like equation.

The approaches discussed in this dissertation derived from Dirac's relativistic Hamiltonian formalism. Dirac's relativistic Hamiltonian formalism is well known for the covariant canonical quantization of relativistic two-body dynamics models such as Todorov's. In a series of papers Dr. H. Crater and Dr. P. Van Alstine have embedded Todorov's effective particle approach in the framework of Dirac's Hamiltonian constraint mechanics for a description of two body systems. Their approach yields manifestly covariant coupled equations. Standard reductions of the Breit equation to a Schrödinger-like equation yield highly singular operators (like δ function and attractive $1/r^3$ potential) that can only be treated perturbatively. In our treatment of the two body Dirac equations of constraint dynamics, we shall find that all the operators are quantum mechanically well defined and we can use nonperturbative techniques(analytic as well as numerical) to obtain solutions of bound state problems and scattering.(A quantum mechanically well defined potential is one no more singular than $-1/4r^2$. If they are not quantum mechanically well defined, they must be treated perturbatively).

Using techniques developed by Dirac to handle constraints in quantum mechanics and the method developed by Crater and Van Alstine, one can derive the two-body Dirac equations for all Lorentz invariant interactions acting together[31]. These include world scalar, four vector and pseudoscalar among others. We can also reduce the two-body Dirac equation to a coupled Schrödinger-like equation. This is very important in the study of nucleon-nucleon interaction, because it allows us to use techniques already developed for the Schrödinger-like systems in nonrelativistic quantum mechanics. Two body Dirac equations of constraint dynamics has been successfully applied in numerical calculations in atomic physics and particle physics. We propose to test this method in nuclear physics in the phase shift analysis of the N-N scattering problems.

3.1 Hamiltonian Formulation Of The Two-Body Problem From Constraint Dynamics

3.1.1 Dirac's Constraint Dynamics For Relativistic Classical Mechanics

Dirac[13] extended Hamiltonian mechanics(which deals with systems in which the conjugate variables, $q's$ and $p's$, are independent of each other) to a mechanics that deal with systems in which the conjugate variables are related by constraints of the form $\phi(q,p) = 0$. For N constraints, we may write

$$\phi_n(q,p) \approx 0 \qquad\qquad n = 1,2,3,\cdots N \tag{3.3}$$

With these constraints the Hamiltonian of the system

$$H = q_n \dot{p}_n - L \tag{3.4}$$

is not unique, we may add any linear combination of the $\phi_n(q,p)$ to the Hamiltonian. We can not distinguish the difference between the Hamiltonian H and the modified or Dirac Hamiltonian

$$\mathcal{H} = H + \lambda_n \phi_n. \tag{3.5}$$

H is called the Legendre Hamiltonian. The λ_n may be the functions of conjugate variables $q's$ and $p's$. We may write the equation of motion for any arbitrary function g of the conjugate variables $q's$ and $p's$ as

$$\dot{g} = [g, \mathcal{H}]. \tag{3.6}$$

Dirac called the conditional equality, $\approx$ as a " weak " equality which mean the constraints $\phi_n \approx 0$ must not be applied before working out the Poisson brackets of above equations.

Dirac called $=$ as a nonconditional equality or a " strong " equality.

$$\dot{g} = [g, \mathcal{H}] = [g, H + \lambda_n \phi_n] = [g, H] + \lambda_n [g, \phi_n] + [g, \lambda_n] \phi_n \qquad (3.7)$$

for $\phi_n \approx 0$, the bracket is worked out first and then we use the " strong" equality $\phi_n = 0$. The weak equality in the above equation means that the sides of the equations differ at most by terms proportional to the constraints $\phi_n \approx 0$.

In the two body system, we have two constraints $\phi_n(q, p) \approx 0, \quad n = 1, 2$. They are the mass shell constraints of the two particles[28], namely

$$\mathcal{H}_1 = p_1^2 + m_1^2 + \Phi_1(x, p_1, p_2) \approx 0, \qquad (3.8)$$

$$\mathcal{H}_2 = p_2^2 + m_2^2 + \Phi_2(x, p_1, p_2) \approx 0, \qquad (3.9)$$

where

$$x = x_1 - x_2. \qquad (3.10)$$

The Dirac Hamiltonian becomes

$$\mathcal{H} = H + \lambda_1 \mathcal{H}_1 + \lambda_2 \mathcal{H}_2. \qquad (3.11)$$

In our case, the Legendre Hamiltonian H is zero. As an example, consider a single particle Lagrangian of the form $L = m\sqrt{-\dot{x}^2}$. The canonical momenta would be $p = -m\,\dot{x}/\sqrt{-\dot{x}^2}$ from which we obtain a single particle constraint $p^2 + m^2 = 0$, so we have $H = q_n \dot{p}_n - L = 0$ and

$$\mathcal{H} = \lambda(p^2 + m^2). \tag{3.12}$$

The Hamiltonian constraints $\mathcal{H}_1$ and $\mathcal{H}_2$ are conserved in terms of a unique evolution parameter τ , so we must obtain

$$\dot{\mathcal{H}_i} = [\mathcal{H}_i, \mathcal{H}] \approx 0, \qquad i = 1, 2. \tag{3.13}$$

Working out this Poisson brackets, we get a new relation between the constraints

$$[\mathcal{H}_1, \mathcal{H}_2] \approx 0 \tag{3.14}$$

This compatibility condition guarantees that with the Dirac Hamiltonian, the system evolves such that the " motion " is constrained to the surface of the mass shell described by the constraints of $\mathcal{H}_1$ and $\mathcal{H}_2$.

3.1.2 Dirac's Constraint Dynamics For Relativistic Quantum Mechanics

Two spin 0 particles[28,29,30] Because the two spinless particle equations and the two spin $\frac{1}{2}$ particle equations are both constraints on the wave equation, we introduce the two spinless particle equation first as it is simple. Both of the two spinless particle equations and two spin $\frac{1}{2}$ particle equations use the same kinematic variables. This facilitates the extension of the two spinless particle case to the two spin $\frac{1}{2}$ particle case.

The relativistic treatment of the two-body problem for spinless particles[29,30] can be written in a way that has the simplicity of the ordinary non-relativistic two-body Schrödinger equation and yet maintains relativistic covariance. Spin and different types of interactions can be carried out in a more complete framework[24,27,31,32]. Dirac extended his idea of the previous section to quantum mechanics by replacing classical constraints $\phi_n(q,p) \approx 0$ with quantum wave equations $\phi_n(q,p) \mid \psi\rangle = 0$, where q and p are conjugate variables. One assumes a generalized mass shell constraint of the form for each individual particles

$$\mathcal{H}_i|\psi\rangle = 0 \qquad \text{for} \qquad i = 1, 2 \tag{3.15}$$

where

$$\mathcal{H}_i = p_i^2 + m_i^2 + \Phi_i, \tag{3.16}$$

and Φ_1 and Φ_2 are two-body interactions dependent on x_{12}. We can construct the total Hamiltonian $\mathcal{H}$ from these constraints by

$$\mathcal{H} = \lambda_1 \mathcal{H}_1 + \lambda_2 \mathcal{H}_2, \tag{3.17}$$

(with λ_i as Lagrange multipliers). In order that each of these constraints be conserved in time we must have

$$[\mathcal{H}_i, \mathcal{H}]|\psi\rangle = i\frac{d\mathcal{H}_i}{d\tau}|\psi\rangle = 0. \tag{3.18}$$

We can obtain

$$[\mathcal{H}_i, \lambda_1\mathcal{H}_1 + \lambda_2\mathcal{H}_2]|\psi\rangle =$$

$$\{[\mathcal{H}_i, \lambda_1]\mathcal{H}_1|\psi\rangle + \lambda_1[\mathcal{H}_i, \mathcal{H}_1]|\psi\rangle + [\mathcal{H}_i, \lambda_2]\mathcal{H}_2|\psi\rangle + \lambda_2[\mathcal{H}_i, \mathcal{H}_2]\}|\psi\rangle = 0.$$

Using Eq.(3.15), the above equation leads to the compatibility condition between the two constraints,

$$[\mathcal{H}_1, \mathcal{H}_2]|\psi\rangle = 0. \tag{3.19}$$

This implies

$$([p_1^2, \Phi_2] + [\Phi_1, p_2^2] + [\Phi_1, \Phi_2])|\psi\rangle = 0. \tag{3.20}$$

Letting

$$\Phi_1 = \Phi_2 = \Phi(x_\perp) \tag{3.21}$$

is the simplest way to satisfy the above equation, which is a kind of relativistic Newton's third law. Here, the transverse coordinate is defined by

$$x_{\nu\perp} = x_{12}^{\mu}(\eta_{\mu\nu} - P_\mu P_\nu/P^2), \tag{3.22}$$

P is the total momentum

$$P = p_1 + p_2. \tag{3.23}$$

Eq.(3.21) leads to

$$[\mathcal{H}_1, \mathcal{H}_2]|\psi\rangle = 2P \cdot \partial_{x_{12}}\Phi(x_\perp)|\psi\rangle = 0, \tag{3.24}$$

and the compatibility condition (3.19) is satisfied.

The Hamiltonian $\mathcal{H}$ determines the dynamics of the two-body system. The equation of motion of the two body system is

$$\mathcal{H}|\psi\rangle = 0. \tag{3.25}$$

This equation includes both the center-of-mass motion and the internal relative motion. To characterize the center-of-mass motion, we note that since the potential Φ depends only on the difference of the two coordinates we have

$$[P, \mathcal{H}]|\psi\rangle = 0. \tag{3.26}$$

(This does not require that $[P, \lambda_i] = 0$ since the $\mathcal{H}_i|\psi\rangle = 0$.) Thus, P is a constant of motion and we can take $|\psi\rangle$ to be an eigenstate characterized by a total momentum P.

We need to separate out the internal relative motion from the center-of-mass motion, introducing a transverse relative momentum p defined by

$$p_1 = \frac{p_1 \cdot P}{P^2} P + p, \tag{3.27}$$

$$p_2 = \frac{p_2 \cdot P}{P^2} P - p, \tag{3.28}$$

where the first term on the right hand side of the above two equations is the projection of each momentum onto the total momentum. The above definition of the relative momentum guarantees the orthogonality of the total momentum and the relative momentum,

$$P \cdot p = 0, \tag{3.29}$$

which follows from taking the scalar product of either equation with P. From Eqs. (3.27) and (3.28) we can rewrite this relative momentum in terms of p_1 and p_2 as

$$p = \frac{\varepsilon_2}{\sqrt{P^2}} p_1 - \frac{\varepsilon_1}{\sqrt{P^2}} p_2 \tag{3.30}$$

where

$$\begin{aligned} \varepsilon_1 &= \frac{p_1 \cdot P}{\sqrt{P^2}} = \frac{P^2 + p_1^2 - p_2^2}{2\sqrt{P^2}} \\ \varepsilon_2 &= \frac{p_2 \cdot P}{\sqrt{P^2}} = \frac{P^2 + p_2^2 - p_1^2}{2\sqrt{P^2}} \end{aligned} \tag{3.31}$$

are the longitudinal components of the momenta in the center-of-momentum system. Using Eqs.(3.15) and (3.21) and taking the difference of the two constraints, we have

$$(p_1^2 - p_2^2)|\psi\rangle = (m_1^2 - m_2^2)|\psi\rangle. \tag{3.32}$$

Thus on these states $|\psi\rangle$ we get

$$\begin{aligned} \varepsilon_1 &= \frac{P^2 + m_1^2 - m_2^2}{2\sqrt{P^2}} \\ \varepsilon_2 &= \frac{P^2 + m_2^2 - m_1^2}{2\sqrt{P^2}}. \end{aligned} \tag{3.33}$$

Using Eqs.(3.27), (3.28), and Eq.(3.29), we can write $\mathcal{H}$ in terms of P and p:

$$\begin{aligned}\mathcal{H}|\psi\rangle &= \{\lambda_1[\varepsilon_1^2 - m_1^2 + p^2 - \Phi(x_\perp)] + \lambda_2[\varepsilon_2^2 - m_2^2 + p^2 - \Phi(x_\perp)]\}|\psi\rangle \\ &= (\lambda_1 + \lambda_2)[b^2(P^2; m_1^2, m_2^2) + p^2 - \Phi(x_\perp)]|\psi\rangle = 0, \end{aligned} \tag{3.34}$$

where

$$b^2(P^2, m_1^2, m_2^2) = \varepsilon_1^2 - m_1^2 = \varepsilon_2^2 - m_2^2 = \frac{1}{4P^2}(P^4 - 2P^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2). \tag{3.35}$$

Equation (3.34) contains both the center-of-mass momentum P and the relative momentum p. We can solve the constraint equation of P and p by the method of the separation of variables. Introducing the bound state eigenvalue w to separate Eq.(3.34) into the following two equations for the center-of-mass motion and the internal motion gives

$$\left\{P^2 - w^2\right\}|\psi\rangle = 0, \tag{3.36}$$

and

$$(\lambda_1 + \lambda_2)\left\{p^2 - \Phi(x_\perp) + b^2(w^2, m_1^2, m_2^2)\right\}|\psi\rangle = 0, \tag{3.37}$$

We have used the first equation on the eigenstate $|\psi\rangle$, so that $b^2(P^2, m_1^2, m_2^2)$ indicates the presence of exact relativistic two-body kinematics:

$$b^2(w^2, m_1^2, m_2^2) = \frac{1}{4w^2}\left\{w^4 - 2w^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2\right\}. \tag{3.38}$$

By this, we mean $p^2 - b^2 = 0$, would imply $w = \sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2}$, which is what we mean by exact kinematics.

Note that if the relative momentum were defined in terms of Eq. (3.33) instead of Eq. (3.31) then we would have

$$p \cdot P|\psi\rangle = 0, \tag{3.39}$$

but not $p \cdot P = 0$, so that $p^2|\psi\rangle = p_\perp^2|\psi\rangle$. In either case the coefficients ε_i are invariant and hence Eq. (3.30) has the same form regardless of which frame it is evaluated in.

When we go to the center-of-momentum system, $p = p_\perp = (0, \mathbf{p})$ and $x_\perp = (0, \mathbf{r})$ (relative energy and time are therefore removed from the problem). We have the equation for the relative motion,

$$\left\{ \frac{\mathbf{p}^2}{2\mu} + \frac{\Phi(\mathbf{r})}{2\mu} - \frac{b^2}{2\mu} \right\} |\psi\rangle = 0, \tag{3.40}$$

where μ is the non-relativistic reduced mass,

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{3.41}$$

We may rewrite Eq. (3.40) into the form of a non-relativistic Schrödinger equation. By renaming $\Phi/2\mu$ as V, and $b^2/2\mu$ as E, Eq. (3.40) becomes

$$\left(\frac{\mathbf{p}^2}{2\mu} + V \right) |\psi\rangle = E|\psi\rangle. \tag{3.42}$$

We can solve the above Schrödinger equation to give the eigenvalue E. Then, from the equation $b^2(w^2, m_1^2, m_2^2) = 2\mu E$, we can solve for w in terms of E and obtain

$$w = \sqrt{2\mu E + m_1^2} + \sqrt{2\mu E + m_2^2}. \tag{3.43}$$

It is easy to show from this that in the nonrelativistic limit, we have the familiar result

$$w = m_1 + m_2 + E. \tag{3.44}$$

If we are only interested in the effect of exact two-body relativistic kinematics with Φ an energy-independent nonrelativistic potential, the bound state eigenvalue w for the relativistic two-body problem is related to the eigenvalue E of the nonrelativistic problem by Eq. (3.43). The potential Φ in relativistic constraint dynamics includes relativistic dynamical corrections as well relativistic kinematical correction. These corrections include dependence of the potential on the CM energy w and on the nature of the interaction. For spinless particles interacting by way of a world scalar interaction S, one finds[30,35,36]

$$\Phi = 2m_w S + S^2 \tag{3.45}$$

where

$$m_w = \frac{m_1 m_2}{w}, \tag{3.46}$$

while for (time-like) vector interactions, one finds[30,35,36,37]

$$\Phi = 2\varepsilon_w A - A^2, \tag{3.47}$$

where

$$\varepsilon_w = \frac{w^2 - m_1^2 - m_2^2}{2w} \tag{3.48}$$

and for combined space-like and time-like vector interactions (that reproduce correct energy spectrum for scalar QED[29])

$$\Phi = 2\varepsilon_w A - A^2 + \vec{\nabla}^2 \log(1 - 2A/w)^{1/2} + [\vec{\nabla} \log(1 - 2A/w)^{1/2}]^2. \tag{3.49}$$

The variables m_w and ε_w (which both approach μ in the nonrelativistic limit) are called the relativistic reduced mass and energy of the fictitious particle of relative motion which were first introduced by Todorov[38] in his quasipotential approach. In the nonrelativistic limit, Φ approaches $2\mu(S + A)$. In the relativistic case, the dynamical corrections to Φ referred to above include both quadratic additions to S and A as well as CM energy dependence through m_w and ε_w.

Eqs.(3.40), (3.42), and (3.43) provide a useful way to obtain the solution of the relativistic two-body problem for spinless particles in scalar and vector interactions. In other works they have been extended to include spin and have been found to give an excellent account of the bound state spectrum of both light and heavy mesons using reasonable input quark potentials.

These ways of putting the invariant potential functions for scalar S and vector A interactions will be used in this dissertation, for the case of two spin one-half particles (see Eq.(5.9) to Eq.(5.13) below), these exact forms are not unique but were motivated by work of Crater and Van Alstine in classical field theory. They play a crucial role in this dissertation since they give us a nonperturbative structure for the form S and A to appear in the equations we use. This structure has been successfully tested in QED(positronium and muonium bound states) and is found to give excellent results when applied to the highly relativistic circumstance of QCD(quark model for mesons). An important question we wish to answer in this dissertation is whether such structures are also valid in the two body nucleon nucleon problem.

Two spin $\frac{1}{2}$ particles We continue our review in this section by introducing the two body Dirac equation of constraint dynamics. We summarize the kinematical variables introduced above to be used for the constraint two body Dirac equations[31]

1) relative position,

$$x_1 - x_2 \tag{3.50}$$

2) total momentum,

$$P = p_1 + p_2 \tag{3.51}$$

3) total c.m. energy

$$w = \sqrt{-P^2} \tag{3.52}$$

4) constituent on-shell c.m. energies,

$$\epsilon_1 = \frac{w^2 + m_1^2 - m_2^2}{2w}, \qquad \epsilon_2 = \frac{w^2 + m_2^2 - m_1^2}{2w} \tag{3.53}$$

5) relativistic reduced mass for fictitious particle of relative motion,

$$m_w = \frac{m_1 m_2}{w} \tag{3.54}$$

6) energy of fictitious particle of relative motion,

$$\epsilon_w = \frac{w^2 - m_1^2 - m_2^2}{2w} \tag{3.55}$$

7) relative momentum

$$p = (\epsilon_2 p_1 - \epsilon_1 p_2)/w. \tag{3.56}$$

On mass shell,

$$p^2 = \epsilon_w^2 - m_w^2 = \frac{w^4 - 2w^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2}{4w^2} = b^2(w) = \epsilon_1^2 - m_1^2 = \epsilon_2^2 - m_2^2 \tag{3.57}$$

In term of these variables,

$$p_1 = \epsilon_1 \hat{P} + p, \qquad p_2 = \epsilon_2 \hat{P} - p \tag{3.58}$$

where $\hat{P} = P/w$.

The Dirac equation for two free spin $\frac{1}{2}$ particles are

$$\mathcal{S}_{10}\psi = (\theta_1 \cdot p_1 + m_1\theta_{51})\psi = 0 \tag{3.59}$$

$$\mathcal{S}_{20}\psi = (\theta_2 \cdot p_2 + m_2\theta_{52})\psi = 0 \tag{3.60}$$

where ψ is the product of the two single-particles Dirac wave equation(these equations are equivalent to the free one body Dirac equation). These two equation are compatible. That is, $[\mathcal{S}_{10}, \mathcal{S}_{20}]\psi = 0$. Substitute p_1 and p_2 to the above two equations and we obtain

$$\mathcal{S}_{10}\psi = (\theta_1 \cdot p + \epsilon_1\theta_1 \cdot \hat{P} + m_1\theta_{51})\psi = 0 \tag{3.61}$$

$$\mathcal{S}_{20}\psi = (-\theta_2 \cdot p + \epsilon_2\theta_2 \cdot \hat{P} + m_2\theta_{52})\psi = 0 \tag{3.62}$$

when expressed in terms of the Todorov variables. The " theta " matrices

$$\theta_i^\mu = i\sqrt{\frac{1}{2}}\gamma_{5i}\gamma_i^\mu, \qquad \mu = 0, 1, 2, 3, \qquad i = 1, 2 \tag{3.63}$$

$$\theta_{5i} = i\sqrt{\frac{1}{2}}\gamma_{5i} \tag{3.64}$$

satisfy the fundamental anticommutation relations

$$[\theta_i^\mu, \theta_i^\nu]_+ = -g^{\mu\nu}, \tag{3.65}$$

$$[\theta_{5i}, \theta_i^{\mu}]_+ = 0, \tag{3.66}$$

$$[\theta_{5i}, \theta_{5i}]_+ = -1, \tag{3.67}$$

The projected " theta " matrices then satisfy

$$\left[\theta_i \cdot \widehat{P}, \theta_i \cdot \widehat{P}\right]_+ = 1, \tag{3.68}$$

$$\left[\theta_i \cdot \widehat{P}, \theta_{i\perp}^{\mu}\right]_+ = 0, \tag{3.69}$$

where

$$\theta_{\nu\perp}^{\mu} = \theta_{i\nu}(\eta^{\mu\nu} - P^{\mu}P^{\nu}). \tag{3.70}$$

Note that defining $\alpha_{i\perp}^{\mu} = 2\theta_i \cdot \hat{P}\ \theta_{i\perp}^{\mu}$, and $\beta_i = 2\theta_i \cdot \hat{P}\ \theta_{5i}$, the above two body Dirac equations become

$$(\alpha_1 \cdot p + \beta_1 m_1)\psi = \epsilon_1 \psi \tag{3.71}$$

$$(-\alpha_2 \cdot p + \beta_2 m_2)\psi = \epsilon_2 \psi \tag{3.72}$$

which have the form of single free particle Dirac equations.

These fundamental anticommutation relations guarantee that the Dirac operators $\mathcal{S}_{10}$ and $\mathcal{S}_{20}$ are the square root of the mass shell operators $-\frac{1}{2}(p_1^2 + m_1^2)$ and $-\frac{1}{2}(p_2^2 + m_2^2)$.

Using above relations, we can get

$$(\mathcal{S}_{10}^2 - \mathcal{S}_{20}^2)\psi = 0 = \frac{1}{2}(p_1^2 + m_1^2 - p_2^2 - m_2^2)\psi \tag{3.73}$$

This leads to the equation

$$P \cdot p\psi = \frac{1}{2}[w(\epsilon_1 - \epsilon_2) - (m_1^2 - m_2^2)]\psi = 0, \tag{3.74}$$

The orthogonality condition $P \cdot p\psi = 0$ is a constraint on the relative momentum p such that it has no time-like component in the center of momentum frame. Just as in the spinless case, where we have the compatibility condition.

$$[\mathcal{H}_1, \mathcal{H}_2]|\psi\rangle = 0 \tag{3.75}$$

so we use here the compatibility condition commmutor to guarantee that the Dirac equations for two spin $\frac{1}{2}$ particles form a compatible set

$$[\mathcal{S}_1, \mathcal{S}_2]|\psi\rangle = 0. \tag{3.76}$$

The attempt to introduce a Lorentz scalar interaction by making the minimal substitutions, which in the case of the single particle Dirac equation (see Eq.(3.1) with $V(r) = \beta S(r)$), become

$$m_i \rightarrow M_i(r) = m_i + S_i \quad i = 1, 2 \tag{3.77}$$

so that

$$\mathcal{S}_1\psi = (\theta_1 \cdot p + \epsilon_1\theta_1 \cdot \hat{P} + M_1\theta_{51})\psi = 0 \tag{3.78}$$

$$\mathcal{S}_2\psi = (-\theta_2 \cdot p + \epsilon_2\theta_2 \cdot \hat{P} + M_2\theta_{52})\psi = 0 \tag{3.79}$$

is not successful. The two equations are not compatible because

$$[\mathcal{S}_1, \mathcal{S}_2]|\psi\rangle = [\theta_1 \cdot p, M_2\theta_{52}]\psi + [\theta_2 \cdot p, M_1\theta_{51}]\psi = -i(\partial M_2\theta_1\theta_{52} + \partial M_1\theta_2\theta_{51})\psi \neq 0 \tag{3.80}$$

In order to make the interactions to meet the compatibility condition, H. Crater and P. Van Alstine found[34] that the two body Dirac equations must be modified in some way. For the scalar interaction this comes from two steps. The first is the introduction of interaction by way of constituent mass potentials M_i related to one another through a relativistic " third law "

$$\partial(M_1^2 - M_2^2) = 0. \tag{3.81}$$

Integrating the " third law ", we obtain

$$M_1^2 - M_2^2 = m_1^2 - m_2^2. \tag{3.82}$$

with the hyperbolic solution

$$M_1 = m_1 ch(L) + m_2 sh(L) \tag{3.83}$$

$$M_2 = m_2 ch(L) + m_1 sh(L) \tag{3.84}$$

Furthermore, as in the spinless case $M_i = M_i(x_\perp)$ depend on the separation variable only through the space-like projection perpendicular to the total momentum, where

$$x_\perp^\mu = \theta_{i\nu}(\eta^{\mu\nu} - \hat{P}^\mu \hat{P}^\nu)(x_1 - x_2)_\nu. \tag{3.85}$$

Without the $x_\perp$ dependence of the potential and the relativistic " third law " condition the constraints would not be compatible. The second step is to add a spin dependent term to the mass potential terms M_i. Now the compatible spin dependent constraint operator $\mathcal{S}_1$ and $\mathcal{S}_2$ become the strongly compatible commuting operators

$$[\mathcal{S}_1, \mathcal{S}_2] = 0 \tag{3.86}$$

with the two body Dirac equations for scalar interactions given by[31]

$$S_1\psi = (\theta_1 \cdot p + \epsilon_1\theta_1 \cdot \hat{P} + M_1\theta_{51} - i\partial L \cdot \theta_2\theta_{52}\theta_{51})\psi = 0 \tag{3.87}$$

$$S_2\psi = (-\theta_2 \cdot p + \epsilon_2\theta_2 \cdot \hat{P} + M_2\theta_{52} + i\partial L \cdot \theta_1\theta_{52}\theta_{51})\psi = 0 \tag{3.88}$$

where from Eq.(3.83) and Eq.(3.84)

$$\partial L = \frac{\partial M_1}{M_2} = \frac{\partial M_2}{M_1}. \tag{3.89}$$

The extra terms ∂L will vanish when the mass of one of the particles become infinite, so we can recover the expected one-body Dirac equation in an external scalar potential(see Eq.(3.1)).

The Dirac constraint operators satisfy[17,18,23]

$$(\mathcal{S}_1^2 - \mathcal{S}_2^2)\psi = -\frac{1}{2}(p_1^2 + m_1^2 - p_2^2 - m_2^2)\psi = -P \cdot p\psi = 0 \tag{3.90}$$

So that the relative momentum remains orthogonal to the momentum in the presence of scalar interactions.

Note that the " effective potential " framework each particles has in the presence of the other in above two body Dirac equations. Crater and Van Alstine showed that they can maintain such a frame work when all interactions are introduced simultaneously.

To do this, Crater and Van Alstine rewrote the two body Dirac equations for scalar interaction in the hyperbolic form[34] below($ch \equiv cosh$, $sh \equiv sinh$)

$$\mathcal{S}_1\psi = (ch(\Delta)\mathbf{S}_1 + sh(\Delta)\mathbf{S}_2)\psi = 0 \tag{3.91}$$

$$\mathcal{S}_2\psi = (ch(\Delta)\mathbf{S}_2 + sh(\Delta)\mathbf{S}_1)\psi = 0 \tag{3.92}$$

where Δ generates the scalar potential terms in the two body Dirac equation equations for scalar interactions provided that

$$\Delta = -\theta_{52}\theta_{51}L(x_\perp). \tag{3.93}$$

Then they replaced the scalar potential by the vector, pseudoscalar, pseudovector, or tensor potentials in above equations. The operators $\mathbf{S_1}$ and $\mathbf{S_2}$ are auxiliary constraints of the form

$$\mathbf{S}_1\psi = (\mathcal{S}_{10}ch(\Delta) + \mathcal{S}_{20}sh(\Delta))\psi = 0 \tag{3.94}$$

$$\mathbf{S}_2\psi = (\mathcal{S}_{20}ch(\Delta) + \mathcal{S}_{10}sh(\Delta))\psi = 0 \tag{3.95}$$

Crater and Van Alstine postulate that above two equation are valid for an arbitrary Lorentz invariant Δ. Then a direct result is that[22,32,34]

$$[P \cdot p, \Delta] = 0 \tag{3.96}$$

or $\Delta(x) = \Delta(x_\perp)$ and

$$P \cdot p\psi = 0. \tag{3.97}$$

This in turn leads to the weak compatibility of both sets of constraint[22,32,34]

$$[\mathcal{S}_1, \mathcal{S}_2]\psi = 0 \tag{3.98}$$

$$[\mathbf{S}_1, \mathbf{S}_2]\psi = 0 \tag{3.99}$$

They considered Δ for the sum of the four “ polar” and four “ axial ” interactions. The four polar interactions are

scalar

$$\Delta_L = -L\theta_{51}\theta_{52} = -\frac{L}{2}\mathcal{O}_1, \mathcal{O}_1 = -\gamma_{51}\gamma_{52}, \tag{3.100}$$

time-like vector

$$\Delta_J = J \hat{P} \cdot\theta_1 \hat{P} \cdot\theta_2 = \mathcal{O}_2\frac{J}{2} = \beta_1\beta_2\frac{J}{2}\mathcal{O}_1, \tag{3.101}$$

space-like vector

$$\Delta_{\mathcal{G}} = \mathcal{G}\theta_{1\perp}\cdot\theta_{2\perp} = \mathcal{O}_3\frac{\mathcal{G}}{2} = \gamma_{1\perp}\gamma_{2\perp}\frac{\mathcal{G}}{2}\mathcal{O}_1, \tag{3.102}$$

tensor(polar)

$$\Delta_{\mathcal{F}} = 4\mathcal{F}\theta_{1\perp}\cdot\theta_{2\perp}\theta_{52}\theta_{51} \hat{P} \cdot\theta_1 \hat{P} \cdot\theta_2 = \mathcal{O}_4\frac{\mathcal{F}}{2} = \alpha_1\cdot\alpha_2\frac{\mathcal{F}}{2}\mathcal{O}_1, \tag{3.103}$$

and their sum

$$\Delta_P = \Delta_L + \Delta_J + \Delta_{\mathcal{G}} + \Delta_{\mathcal{F}} \tag{3.104}$$

can replace the Δ in the two body Dirac equations for scalar without any change.

For time like vector interactions

$$\Delta_J = \frac{\mathcal{O}_2 J(x_\perp)}{2} = \frac{\gamma_1\cdot \hat{P} \gamma_2\cdot \hat{P} J(x_\perp)}{2}\mathcal{O}_1 \tag{3.105}$$

where $\mathcal{O}_2 = 2\theta_1\cdot\hat{P}\theta_2\cdot\hat{P}$.

For space like vector interactions

$$\Delta_{\mathcal{G}} = \frac{\mathcal{O}_3 \mathcal{G}(x_\perp)}{2} = \frac{\gamma_{1\perp} \cdot \gamma_{2\perp} \mathcal{G}(x_\perp)}{2} \mathcal{O}_1 \tag{3.106}$$

where $\mathcal{O}_3 = 2\gamma_{1\perp} \cdot \gamma_{2\perp}$. A matrix amplitude proportional to $\gamma_1^\mu \cdot \gamma_{2\mu}$ corresponding to an electromagnetic-like interaction would indicate that $J = -\mathcal{G}$[25].

The four " axial " interactions are

pseudoscalar

$$\Delta_C = \mathcal{E}_1 \frac{C}{2} = -\gamma_{51}\gamma_{52} \frac{C}{2} \mathcal{O}_1, \tag{3.107}$$

time-like pseudovector

$$\Delta_H = -2H \hat{P} \cdot \theta_1 \hat{P} \cdot \theta_2 \theta_{51} \theta_{52} = -\mathcal{E}_2 \frac{H}{2} = \beta_1 \gamma_{51} \beta_2 \gamma_{52} \frac{H}{2} \mathcal{O}_1, \tag{3.108}$$

space-like pseudovector

$$\Delta_I = -2I \theta_{1\perp} \cdot \theta_{2\perp} \theta_{51} \theta_{52} = -\mathcal{E}_3 \frac{I}{2} = -\gamma_{51} \gamma_{1\perp} \cdot \gamma_{52} \gamma_{2\perp} \frac{I}{2} \mathcal{O}_1, \tag{3.109}$$

tensor(axial)

$$\Delta_Y = -2Y \theta_{1\perp} \cdot \theta_{2\perp} \hat{P} \cdot \theta_1 \hat{P} \cdot \theta_2 = -\mathcal{E}_4 \frac{Y}{2} = -\sigma_1 \cdot \sigma_2 \frac{Y}{2} \mathcal{O}_1, \tag{3.110}$$

$$\Delta_a = \Delta_C + \Delta_H + \Delta_I + \Delta_Y \tag{3.111}$$

Crater and Van Alstine found that these would be used in Eq.(3.91) , Eq.(3.92), Eq.(3.94), Eq.(3.95), but with the $sh(\Delta_a)$ terms in Eq.(3.91) and Eq.(3.92) appearing with a negative sign instead of the plus sign as is the case polar interactions[34] . There is no sign change in Eq.(3.94) and Eq.(3.95) for Δ_a.

For systems with both polar and axial interactions[34], one uses $\Delta_p - \Delta_a$ to replace Δ in Eq.(3.91) and Eq.(3.92), and $\Delta_p + \Delta_a$ replace the Δ in Eq.(3.94) and Eq.(3.95). L, J, $\mathcal{G}$, $\mathcal{F}$, C, H, I, Y are arbitrary invariant functions of $x_\perp$. In this dissertation, we include only mesons corresponding to the interactions L, J, $\mathcal{G}(J = -\mathcal{G})$, C. Thus we are ignoring tensor and pseudovector mesons, limiting ourselves to vector, scalar and pseudoscalar mesons, all having masses less than or about 1000 MeV.

3.1.3 The General Constraints $\mathcal{S}_1$ and $\mathcal{S}_2$

Crater and Van Alstine have derived[34] the " external potential " forms of the constraint two body Dirac equations for each of the eight interaction matrices, Δ_L, Δ_J, $\Delta_\mathcal{G}$, $\Delta_\mathcal{F}$, Δ_C, Δ_H, Δ_I, Δ_Y.

Polar constraints from the general hyperbolic form: For each of the polar interaction Δ_L, Δ_J, $\Delta_\mathcal{G}$, $\Delta_\mathcal{F}$ or any combination of them, a generalization of Eq.(3.91) and Eq.(3.92) for " external potential " forms of the constraints $\mathcal{S}_1$ and $\mathcal{S}_2$have the following relation to the auxiliary constraints $\mathbf{S}_1$ and $\mathbf{S}_2$:

$$\mathcal{S}_1\psi = (ch\Delta_p\mathbf{S}_1 + sh\Delta_p\mathbf{S}_2)\psi = 0 \tag{3.112}$$

$$\mathcal{S}_2\psi = (ch\Delta_p\mathbf{S}_2 + sh\Delta_p\mathbf{S}_1)\psi = 0 \qquad (3.113)$$

using the generalization of these auxiliary constraints one finds[34] that $\mathcal{S}_1$ becomes

$$\begin{aligned} \mathcal{S}_1\psi &= (\mathcal{S}_{10} + ch\Delta_p[\mathcal{S}_{10}, ch\Delta_p]_- + ch\Delta_p[\mathcal{S}_{20}, sh\Delta_p]_+ \\ &\quad + sh\Delta_p[\mathcal{S}_{20}, ch\Delta_p]_- + sh\Delta_p[\mathcal{S}_{10}, sh\Delta_p]_+)\psi \end{aligned} \qquad (3.114)$$

Working out the commutation and anticommutation brackets in above equation, Crater and Van Alstine found the generalized expression for Δ_p

$\mathrm{S}_1\psi = (\mathcal{S}_{10} + i\theta_2 \cdot \partial\Delta_p$

$$- \{[ch\Delta_p, \theta_1^\mu]_- sh\Delta_p - [ch\Delta_p, \theta_2^\mu]_- ch\Delta_p - [sh\Delta_p, \theta_2^\mu]_+ sh\Delta_p + [sh\Delta_p, \theta_1^\mu]_+ ch\Delta_p\}\partial_\mu\Delta_p$$

$$+ \{ch\Delta_p[\theta_1^\mu, ch\Delta_p]_- - ch\Delta_p[\theta_2^\mu, sh\Delta_p]_+ - sh\Delta_p[\theta_2^\mu, ch\Delta_p]_- + sh\Delta_p[\theta_1^\mu, sh\Delta_p]_+\}p_\mu$$

$$+ ch\Delta_p[\epsilon_1\theta_1\cdot \hat{P}, ch\Delta_p]_- + ch\Delta_p[\epsilon_2\theta_2\cdot \hat{P}, sh\Delta_p]_+ + sh\Delta_p[\epsilon_2\theta_2\cdot \hat{P}, ch\Delta_p]_-$$

$$+ sh\Delta_p[\epsilon_1\theta_1\cdot \hat{P}, sh\Delta_p]_+ + ch\Delta_p[m_1\theta_{51}, ch\Delta_p]_- + ch\Delta_p[m_2\theta_{52}, sh\Delta_p]_+$$

$+ sh\Delta_p[m_2\theta_{52}, ch\Delta_p]_- + sh\Delta_p[m_1\theta_{51}, sh\Delta_p]_+)\psi,$

One can obtain this expression for $\mathcal{S}_2$ by a similar method.

Axial constraints from the general hyperbolic form: For each of the axial interaction Δ_C, Δ_H, Δ_I, Δ_Y or any combination of them, a generalization of Eq. (3.91) and Eq.(3.92) for " external potential " forms of the constraints $\mathcal{S}_1$ and $\mathcal{S}_2$have the following relation to the auxiliary constraints $\mathbf{S}_1$ and $\mathbf{S}_2$[34] :

$$\mathcal{S}_1\psi = (ch\Delta_a\mathbf{S}_1 - sh\Delta_a\mathbf{S}_2)\psi = 0 \tag{3.115}$$

$$\mathcal{S}_2\psi = (ch\Delta_a\mathbf{S}_2 - sh\Delta_a\mathbf{S}_1)\psi = 0 \tag{3.116}$$

The only difference with the polar interactions is the plus sign change to minus sign. In terms of the axial invariant matrix Δ_a Crater and Van Alstine found $\mathcal{S}_1$ [34]

$$\begin{aligned}
\mathcal{S}_1\psi &= (\mathcal{S}_{10} + ch\Delta_a[\mathcal{S}_{10}, ch\Delta_a]_- + ch\Delta_a[\mathcal{S}_{20}, sh\Delta_a]_- \\
&\quad + sh\Delta_a[\mathcal{S}_{20}, ch\Delta_a]_- - sh\Delta_a[\mathcal{S}_{10}, sh\Delta_a]_-)\psi \\
&\quad + sh\Delta_a[\mathcal{S}_{20}, ch\Delta_a]_- + sh\Delta_a[\mathcal{S}_{10}, sh\Delta_a]_+)\psi \qquad (3.117)
\end{aligned}$$

working out the commutation and anticommutation brackets in above equation, yields the generalized expression for $\mathcal{S}_1$

$$\mathrm{S}_1\psi = (\mathcal{S}_{10} + i\theta_2 \cdot \partial\Delta_a$$

$$-\{[ch\Delta_a, \theta_1^{\mu}]_- sh\Delta_a - [ch\Delta_a, \theta_2^{\mu}]_- ch\Delta_a + [sh\Delta_a, \theta_2^{\mu}]_- sh\Delta_a - [sh\Delta_a, \theta_1^{\mu}]_- ch\Delta_a\}\partial_\mu\Delta_a$$

$$+\{ch\Delta_a[\theta_1^{\mu}, ch\Delta_a]_- - ch\Delta_a[\theta_2^{\mu}, sh\Delta_a]_- + sh\Delta_a[\theta_2^{\mu}, ch\Delta_a]_- - sh\Delta_a[\theta_1^{\mu}, sh\Delta_a]_-\}p_\mu$$

$$+ ch\Delta_a[\epsilon_1\theta_1 \cdot \hat{P}, ch\Delta_a]_- + ch\Delta_a[\epsilon_2\theta_2 \cdot \hat{P}, sh\Delta_a]_- - sh\Delta_a[\epsilon_2\theta_2 \cdot \hat{P}, ch\Delta_a]_-$$

$$- sh\Delta_a[\epsilon_1\theta_1 \cdot \hat{P}, sh\Delta_a]_- + ch\Delta_a[m_1\theta_{51}, ch\Delta_a]_- + ch\Delta_a[m_2\theta_{52}, sh\Delta_a]_-$$

$$- sh\Delta_a[m_2\theta_{52}, ch\Delta_a]_- - sh\Delta_a[m_1\theta_{51}, sh\Delta_a]_-)\psi$$

One can obtain this expression for $\mathcal{S}_2$ by a similar method.

The derivations for above two equations are very lengthy, for the simple case of individual interactions, many of the commutators vanish, the calculation are fairly short. Crater and Van Alstine found that the following two body Dirac equations are the results of the calculation for the four polar interaction Δ_L, Δ_J, $\Delta_{\mathcal{G}}$, $\Delta_{\mathcal{F}}$ and the four axial interactions Δ_C, Δ_H, Δ_I, Δ_Y acting independently.

Polar Interactions $\Delta_p's$

Scalar interaction: $\Delta = \Delta_L$

$$S_1\psi = (\theta_1 \cdot p + \epsilon_1\theta_1 \cdot \hat{P} + m_1 ch(L\mathcal{O}_1)\theta_{51} - m_2 sh(L\mathcal{O}_1)\theta_{52} - i\theta_2 \cdot \frac{\partial L}{2}\mathcal{O}_1)\psi = 0 \quad (3.118)$$

$$S_2\psi = (-\theta_2 \cdot p + \epsilon_2\theta_2 \cdot \hat{P} + m_2 ch(L\mathcal{O}_1)\theta_{52} - m_1 sh(L\mathcal{O}_1)\theta_{51} + i\theta_1 \cdot \frac{\partial L}{2}\mathcal{O}_1)\psi = 0 \quad (3.119)$$

Time-like vector interaction: $\Delta = \Delta_J$

$$S_1\psi = (\theta_1 \cdot p + \epsilon_1 ch(J\mathcal{O}_2)\theta_1 \cdot \hat{P} + \epsilon_2 sh(J\mathcal{O}_2)\theta_2 \cdot \hat{P} + m_1\theta_{51} + i\theta_2 \cdot \frac{\partial J}{2}\mathcal{O}_2)\psi = 0 \quad (3.120)$$

$$S_2\psi = (-\theta_2 \cdot p + \epsilon_2 ch(J\mathcal{O}_2)\theta_2 \cdot \hat{P} + \epsilon_1 sh(J\mathcal{O}_2)\theta_1 \cdot \hat{P} + m_2\theta_{52} - i\theta_1 \cdot \frac{\partial J}{2}\mathcal{O}_2)\psi = 0 \quad (3.121)$$

Space-like vector interaction: $\Delta = \Delta_{\mathcal{G}}$

$$S_1\psi = (\exp(\mathcal{G})\theta_1 \cdot p + \epsilon_1\theta_1 \cdot \hat{P} + m_1\theta_{51} + i\exp(\mathcal{G})\theta_2 \cdot \frac{\partial \mathcal{G}}{2}\mathcal{O}_3)\psi = 0 \quad (3.122)$$

$$S_2\psi = (-\exp(\mathcal{G})\theta_2 \cdot p + \epsilon_2\theta_2 \cdot \hat{P} + m_2\theta_{52} - i\exp(\mathcal{G})\theta_1 \cdot \frac{\partial \mathcal{G}}{2}\mathcal{O}_3)\psi = 0 \quad (3.123)$$

Polar tensor interaction: $\Delta = \Delta_{\mathcal{F}}$

$$S_1\psi = (\exp(\mathcal{F}\mathcal{E}_2)\theta_1 \cdot p + \epsilon_1 ch(\mathcal{F}\mathcal{O}_4)\theta_1 \cdot \hat{P} + \epsilon_2 sh(\mathcal{F}\mathcal{O}_4)\theta_2 \cdot \hat{P}$$

$$+ m_1 ch(\mathcal{F}\mathcal{O}_4)\theta_{51} + m_2 sh(\mathcal{F}\mathcal{O}_4)\theta_{52} + i\exp(\mathcal{F}\mathcal{E}_2)\theta_2 \cdot \frac{\partial \mathcal{F}}{2}\mathcal{O}_4)\psi = 0 \quad (3.124)$$

$$S_2\psi = (-\exp(\mathcal{F}\mathcal{E}_2)\theta_2 \cdot p + \epsilon_2 ch(\mathcal{F}\mathcal{O}_4)\theta_2 \cdot \hat{P} + \epsilon_1 sh(\mathcal{F}\mathcal{O}_4)\theta_1 \cdot \hat{P}$$

$$+m_2 ch(\mathcal{F}\mathcal{O}_4)\theta_{52} + m_1 sh(\mathcal{F}\mathcal{O}_4)\theta_{51} - i\exp(\mathcal{F}\mathcal{E}_2)\theta_1 \cdot \frac{\partial\mathcal{F}}{2}\mathcal{O}_4)\psi = 0 \tag{3.125}$$

Axial Interactions $\Delta_a's$

Pseudoscalar interaction: $\Delta = \Delta_C$

$$S_1\psi = (\theta_1 \cdot p + \epsilon_1\theta_1 \cdot \hat{P} + m_1\theta_{51} + i\theta_2 \cdot \frac{\partial C}{2}\mathcal{E}_1)\psi = 0 \tag{3.126}$$

$$S_2\psi = (-\theta_2 \cdot p + \epsilon_2\theta_2 \cdot \hat{P} + m_2\theta_{52} - i\theta_1 \cdot \frac{\partial C}{2}\mathcal{E}_1)\psi = 0 \tag{3.127}$$

Time-like pseudovector interaction: $\Delta = \Delta_H$

$$S_1\psi = (\theta_1 \cdot p + \epsilon_1 ch(H\mathcal{E}_2)\theta_1 \cdot \hat{P} + \epsilon_2 sh(H\mathcal{E}_2)\theta_2 \cdot \hat{P}$$

$$+m_1 ch(H\mathcal{E}_1)\theta_{51} + m_2 sh(H\mathcal{E}_1)\theta_{52} - i\theta_2 \cdot \frac{\partial H}{2}\mathcal{E}_2)\psi = 0 \tag{3.128}$$

$$S_2\psi = (-\theta_2 \cdot p + \epsilon_2 ch(H\mathcal{E}_2)\theta_2 \cdot \hat{P} + \epsilon_1 sh(H\mathcal{E}_2)\theta_1 \cdot \hat{P}$$

$$+m_2 ch(H\mathcal{E}_1)\theta_{52} + m_1 sh(H\mathcal{E}_1)\theta_{51} + i\theta_1 \cdot \frac{\partial H}{2}\mathcal{E}_2)\psi = 0 \tag{3.129}$$

Space-like pseudovector interaction: $\Delta = \Delta_I$

$$S_1\psi = (\exp(I\mathcal{O}_1)\theta_1 \cdot p + \epsilon_1\theta_1 \cdot \hat{P} + m_1 ch(I\mathcal{E}_3)\theta_{51} + m_2 sh(I\mathcal{E}_3)\theta_{52} - i\exp(I\mathcal{O}_1)\theta_2 \cdot \frac{\partial I}{2}\mathcal{E}_3)\psi = 0 \tag{3.130}$$

$$S_2\psi = (-\exp(I\mathcal{O}_1)\theta_2 \cdot p + \epsilon_2\theta_2 \cdot \hat{P} + m_2 ch(I\mathcal{E}_3)\theta_{52} + m_1 sh(I\mathcal{E}_3)\theta_{51} + i\exp(I\mathcal{O}_1)\theta_1 \cdot \frac{\partial I}{2}\mathcal{E}_3)\psi = 0 \tag{3.131}$$

Pseudotensor interaction: $\Delta = \Delta_I$

$$S_1\psi =$$

$$(\exp(Y\mathcal{O}_2)\theta_1 \cdot p + \epsilon_1 ch(Y\mathcal{E}_4)\theta_1 \cdot \hat{P} + \epsilon_2 sh(Y\mathcal{E}_4)\theta_2 \cdot \hat{P} + m_1\theta_{51} - i\exp(Y\mathcal{O}_2)\theta_2 \cdot \frac{\partial Y}{2}\mathcal{E}_4)\psi = 0 \tag{3.132}$$

$$S_2\psi =$$

$$(-\exp(Y\mathcal{O}_2)\theta_2 \cdot p + \epsilon_2 ch(Y\mathcal{E}_4)\theta_2 \cdot \hat{P} + \epsilon_1 sh(Y\mathcal{E}_4)\theta_1 \cdot \hat{P} + m_2\theta_{52} + i\exp(Y\mathcal{O}_2)\theta_1 \cdot \frac{\partial Y}{2}\mathcal{E}_4)\psi = 0 \tag{3.133}$$

Combining Polar and Axial Interactions The preceding commutation relations for $\mathcal{S}_1$and $\mathcal{S}_2$ are also suitable for the combinations of the polar interactions $\Delta_p = \Delta_L + \Delta_J + \Delta_{\mathcal{G}} + \Delta_{\mathcal{F}}$ and axial interactions $\Delta_a = \Delta_C + \Delta_H + \Delta_I + \Delta_Y$. To combine the eight interaction together, we need to find the general expressions for $\mathcal{S}_1$and $\mathcal{S}_2$ that are valid for the that is valid for the mixed symmetries of the general invariant matrix $\Delta = \Delta_a + \Delta_p$. The resulting expressions for $\mathcal{S}_1$and $\mathcal{S}_2$ are in very complicated form[31].

From these complicated expressions of $\mathcal{S}_1$and $\mathcal{S}_2$, the complete hyperbolic constraint two body Dirac equations for all the eight interactions acting together was obtained by Long and Crater[31]

$$\mathcal{S}_1\psi =$$

$$\{\exp(\mathcal{G}+\mathcal{F}\mathcal{E}_2+I\mathcal{O}_1+Y\mathcal{O}_2)[\theta_1\cdot p-\frac{i}{2}\theta_2\cdot\partial(L\mathcal{O}_1-J\mathcal{O}_2-\mathcal{G}\mathcal{O}_3-\mathcal{F}\mathcal{O}_4-C\mathcal{E}_1+H\mathcal{E}_2+I\mathcal{E}_3+Y\mathcal{E}_4)]$$

$$+\epsilon_1\mathrm{ch}(J\mathcal{O}_2+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+Y\mathcal{E}_4)\theta_1\cdot\hat{P}+\epsilon_2\mathrm{sh}(J\mathcal{O}_2+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+Y\mathcal{E}_4)\theta_2\cdot\hat{P}$$

$$+m_1\mathrm{ch}(-L\mathcal{O}_1+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+I\mathcal{E}_3)\theta_{51}+m_2\mathrm{sh}(-L\mathcal{O}_1+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+I\mathcal{E}_3)\theta_{52}\}\psi=0. \tag{3.134}$$

$$\mathcal{S}_2\psi =$$

$$\{-\exp(\mathcal{G}+\mathcal{F}\mathcal{E}_2+I\mathcal{O}_1+Y\mathcal{O}_2)[\theta_2\cdot p-\frac{i}{2}\theta_1\cdot\partial(L\mathcal{O}_1-J\mathcal{O}_2-\mathcal{G}\mathcal{O}_3-\mathcal{F}\mathcal{O}_4-C\mathcal{E}_1+H\mathcal{E}_2+I\mathcal{E}_3+Y\mathcal{E}_4)]$$

$$+\epsilon_1\mathrm{sh}(J\mathcal{O}_2+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+Y\mathcal{E}_4)\theta_1\cdot\hat{P}+\epsilon_2\mathrm{ch}(J\mathcal{O}_2+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+Y\mathcal{E}_4)\theta_2\cdot\hat{P}$$

$$+m_1\mathrm{sh}(-L\mathcal{O}_1+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+I\mathcal{E}_3)\theta_{51}+m_2\mathrm{ch}(-L\mathcal{O}_1+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+I\mathcal{E}_3)\theta_{52}\}\psi=0. \tag{3.135}$$

One can obtain all the Eq(3.118) to Eq(3.133) from above two body Dirac equations by imposing some limits. For example, let all the interaction vanish except C , above equations will collapse to the pseudoscalar constraint equations. In the case of combined scalar, time-like, space-like and pseudoscalar interactions,which we use in this dissertation, the two body Dirac equations with

$$\Delta = \Delta_J + \Delta_L + \Delta_{\mathcal{G}} + \Delta_C \tag{3.136}$$

are

$$\mathcal{S}_1\psi = (G\theta_1\cdot p + E_1\theta_1\cdot\hat{P} + M_1\theta_{51} + i\frac{G}{2}(\theta_2\cdot\partial\mathcal{G}\mathcal{O}_3 + \theta_2\cdot\partial J\mathcal{O}_2 - \theta_2\cdot\partial L\mathcal{O}_1 + \theta_2\cdot\partial C\mathcal{E}_1))\psi = 0 \tag{3.137}$$

$$\mathcal{S}_2\psi = (-G\theta_2\cdot p + E_2\theta_2\cdot\hat{P} + M_2\theta_{52} - i\frac{G}{2}(\theta_1\cdot\partial\mathcal{G}\mathcal{O}_3 + \theta_1\cdot\partial J\mathcal{O}_2 - \theta_1\cdot\partial L\mathcal{O}_1 + \theta_1\cdot\partial C\mathcal{E}_1))\psi = 0. \tag{3.138}$$

where

$$M_1 = m_1\mathrm{ch}(L) + m_2\mathrm{sh}(L), \tag{3.139}$$

$$M_2 = m_2\mathrm{ch}(L) + m_1\mathrm{sh}(L), \tag{3.140}$$

$$E_1 = \epsilon_1\mathrm{ch}(J) + \epsilon_2\mathrm{sh}(J), \tag{3.141}$$

$$E_2 = \epsilon_2\mathrm{ch}(J) + \epsilon_1\mathrm{sh}(J), \tag{3.142}$$

$$G = e^{\mathcal{G}}, \tag{3.143}$$

The above two body Dirac equations(without pseudoscalar interactions) have been tested successfully in quark model calculations of the meson spectra[19,20,21,26].

In the limit $m_1 \to \infty$ (or $m_2 \to \infty$), this means one of the particles become infinitely massive, the extra terms $\partial\mathcal{G}$, $\partial J, \partial L$ and ∂C in Eq.(3.137) and Eq.(3.138) vanish, one recovers the expected one body Dirac equation in an external potential.

3.2 Pauli Reduction

Now, one can use the complete hyperbolic constraint two body Dirac equations Eq.(3.134) and Eq.(3.135), to derive the Schrödinger-like eigenvalue equation for the combined interactions: $L(x_\perp), J(x_\perp), H(x_\perp), C(x_\perp), \mathcal{G}(x_\perp), \mathcal{F}(x_\perp), I(x_\perp), Y(x_\perp)$[31] . In this dissertation, however, we include only mesons corresponding to the interactions L, J, $\mathcal{G}(J = -\mathcal{G})$, C, thus limiting ourselves to vector, scalar and pseudoscalar mesons. The basic method use here has some similarities to the reduction of the single particle Dirac equation to a Schrödinger-like form (Pauli-reduction) and related work by Sazdjian[22,27].

The wave function ψ appearing in the two-body Dirac equations Eq.(3.134) and Eq.(3.135) is a Dirac spinor written

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \tag{3.144}$$

where each ψ_i is itself a four component spinor. The wavefunction ψ has a total of sixteen components and the matrices $\mathcal{O}_i$'s, $\mathcal{E}_i$'s are all sixteen by sixteen. In terms of the gamma matrices we use the Dirac representations with block forms

$$\beta_1 = \begin{pmatrix} 1_8 & 0 \\ 0 & -1_8 \end{pmatrix}, \qquad \gamma_{51} = \begin{pmatrix} 0 & 1_8 \\ 1_8 & 0 \end{pmatrix}, \qquad \beta_1\gamma_{51} \equiv \rho_1 = \begin{pmatrix} 0 & 1_8 \\ -1_8 & 0 \end{pmatrix}$$

$$\beta_2 = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1_4 & 0 \\ 0 & -1_4 \end{pmatrix}$$

$$\gamma_{52} = \begin{pmatrix} \gamma_5 & 0 \\ 0 & \gamma_5 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 0 & 1_4 \\ 1_4 & 0 \end{pmatrix}$$

$$\beta_2\gamma_{52} \equiv \rho_2 = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}, \qquad \rho = \begin{pmatrix} 0 & 1_4 \\ -1_4 & 0 \end{pmatrix}$$

$$\beta_1\gamma_{51}\gamma_{52} = \begin{pmatrix} 0 & \gamma_5 \\ -\gamma_5 & 0 \end{pmatrix}, \qquad \gamma_{51}\gamma_{52} = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}, \qquad \beta_2\gamma_{51}\gamma_{52} = \begin{pmatrix} 0 & \rho \\ \rho & 0 \end{pmatrix}$$

$$\mathbf{\Sigma}_i^\mu = \gamma_{5i}\beta_i\gamma_{\perp i}^\mu, \; i = 1, 2.$$

$\mathbf{\Sigma}_i^\mu$ are four-vector generalizations of the Pauli matrices of particles one and two. In the c.m. frame the time component is zero and the spatial components are the usual Pauli matrices for each particle. Here, we include only mesons corresponding to the interactions L, J, $\mathcal{G}(J = -\mathcal{G})$, C. In terms of these matrices we rewrite Eq.(3.134) and Eq.(3.135) by multiplying the first by $\sqrt{2}i\beta_1$ and the second by $\sqrt{2}i\beta_2$ yielding[31]

$$[T_1(\beta_1\beta_2) + U_1(\beta_1\beta_2)\gamma_{51}\gamma_{52}]\psi = (E_1 + M_1\beta_1)\gamma_{51}\psi \tag{3.145}$$

$$-[T_2(\beta_1\beta_2) + U_2(\beta_1\beta_2)\gamma_{51}\gamma_{52}]\psi = (E_2 + M_2\beta_2)\gamma_{52}\psi \tag{3.146}$$

in which the kinetic and recoil terms are

$$T_1(\beta_1\beta_2) = exp(\mathcal{G})[\mathbf{\Sigma}_1 \cdot p - \frac{i}{2}\beta_1\beta_2(\mathbf{\Sigma}_2 \cdot \partial(-C + \mathcal{G}\beta_1\beta_2\mathbf{\Sigma}_1 \cdot \mathbf{\Sigma}_2)]$$

$$T_2(\beta_1\beta_2) = exp(\mathcal{G})[\mathbf{\Sigma}_2 \cdot p - \frac{i}{2}\beta_1\beta_2(\mathbf{\Sigma}_1 \cdot \partial(-C + \mathcal{G}\beta_1\beta_2\mathbf{\Sigma}_1 \cdot \mathbf{\Sigma}_2)]$$

$$U_1(\beta_1\beta_2) = exp(\mathcal{G})[-\frac{i}{2}\beta_1\beta_2\mathbf{\Sigma}_2 \cdot \partial(J\beta_1\beta_2 - L)]$$

$$U_2(\beta_1\beta_2) = exp(\mathcal{G})[-\frac{i}{2}\beta_1\beta_2\mathbf{\Sigma}_1 \cdot \partial(J\beta_1\beta_2 - L)]$$

while the timelike and scalar potentials are

$$E_1 = \epsilon_1 \mathrm{ch}(J) + \epsilon_2 \mathrm{sh}(J)$$

$$E_2 = \epsilon_2 \mathrm{ch}(J) + \epsilon_1 \mathrm{sh}(J)$$

$$M_1 = m_1 \mathrm{ch}(L) + m_2 \mathrm{sh}(L)$$

$$M_2 = m_2 \mathrm{ch}(L) + m_1 \mathrm{sh}(L).$$

The end result of the matrix multiplication in Eq(3.145) and Eq(3.146) is a set of eight simultaneous equations for the Dirac spinors $\psi_1, \psi_2, \psi_3, \psi_4$. In an arbitrary frame, the result of the matrix calculation produces the eight simultaneous equations ($\sigma_i^\mu \psi \to \mathbf{\Sigma}_i^\mu \psi_{1,2,3,4}$):

$$T_1(+1)\psi_1 + U_1(+1)\psi_4 = (E_1 + M_1)\psi_3 \tag{3.147}$$

$$T_1(-1)\psi_2 + U_1(-1)\psi_3 = (E_1 + M_1)\psi_4 \tag{3.148}$$

$$T_1(-1)\psi_3 + U_1(-1)\psi_2 = (E_1 - M_1)\psi_1 \tag{3.149}$$

$$T_1(+1)\psi_4 + U_1(+1)\psi_1 = (E_1 - M_1)\psi_2 \tag{3.150}$$

$$-T_2(+1)\psi_1 - U_2(+1)\psi_4 = (E_2 + M_2)\psi_2 \tag{3.151}$$

$$-T_2(-1)\psi_2 - U_2(-1)\psi_3 = (E_2 - M_2)\psi_1 \tag{3.152}$$

$$-T_2(-1)\psi_3 - U_2(-1)\psi_2 = (E_2 + M_2)\psi_4 \tag{3.153}$$

$$-T_2(+1)\psi_4 - U_2(+1)\psi_1 = (E_2 - M_2)\psi_3 \tag{3.154}$$

One now reduces the above set of eight equations to a second order Schrödinger-like equation by a process of substitution and elimination using the combination of the four Dirac-spinors given below[31]:

$$\phi_\pm \equiv \psi_1 \pm \psi_4,$$

$$\chi_\pm \equiv \psi_2 \pm \psi_3.$$

Eq.(3.147) plus Eq.(3.150) yields

$$D_1^{++}\phi_+ = E_1\chi_+ - M_1\chi_- \tag{3.155}$$

Eq.(3.151) plus Eq.(3.154) yields

$$-D_2^{++}\phi_+ = E_2\chi_+ + M_2\chi_- \tag{3.156}$$

Eq.(3.148) plus Eq.(3.149) yields

$$D_1^{-+}\chi_+ = E_1\phi_+ - M_1\phi_- \tag{3.157}$$

Eq.(3.148) minus Eq.(3.149) yields

$$D_1^{--}\chi_- = -E_1\phi_- + M_1\phi_+ \tag{3.158}$$

with the following definitions for the kinetic-recoil terms

$$\begin{aligned} D_1^{++} \equiv T_1(+1) \quad &+ \quad U_1(+1) = \\ &\mathrm{e}^{\mathcal{G}}[\sigma_1 \cdot p - \frac{i}{2}\sigma_2 \cdot \partial[-C + J - L + \mathcal{G}\sigma_1 \cdot \sigma_2]] \end{aligned} \tag{3.159}$$

$$\begin{aligned} D_2^{++} \equiv T_2(+1) \quad &+ \quad U_2(+1) = \\ &\mathrm{e}^{\mathcal{G}}[\sigma_2 \cdot p - \frac{i}{2}\sigma_1 \cdot \partial[-C + J - L + \mathcal{G}\sigma_1 \cdot \sigma_2]] \end{aligned} \tag{3.160}$$

$$\begin{aligned} D_1^{-+} \equiv T_1(-1) \quad &+ \quad U_1(-1) = \\ &\mathrm{e}^{\mathcal{G}}[\sigma_1 \cdot p + \frac{i}{2}\sigma_2 \cdot \partial[-C - J - L - \mathcal{G}\sigma_1 \cdot \sigma_2]] \end{aligned} \tag{3.161}$$

$$\begin{aligned} D_1^{--} \equiv T_1(-1) \quad &- \quad U_1(-1) = \\ &\mathrm{e}^{\mathcal{G}}[\sigma_1 \cdot p + \frac{i}{2}\sigma_2 \cdot \partial[-C + J + L - \mathcal{G}\sigma_1 \cdot \sigma_2]]. \end{aligned} \tag{3.162}$$

Solve Eq.(3.155) and Eq.(3.156) for χ_+ and χ_-.

$$\chi_+ = \frac{1}{\mathcal{D}}(M_2 D_1^{++} - M_1 D_2^{++})\phi_+ \tag{3.163}$$

$$\chi_- = -\frac{1}{\mathcal{D}}(E_2 D_1^{++} + E_1 D_2^{++})\phi_+ \tag{3.164}$$

in which

$$\mathcal{D} \equiv E_1 M_2 + E_2 M_1. \tag{3.165}$$

Solve Eq.(3.157) and Eq.(3.158) for ϕ_+.

$$E_1 D_1^{-+}\chi_+ - M_1 D_1^{--}\chi_- = \mathcal{B}^2\phi_+ \tag{3.166}$$

in which

$$\mathcal{B}^2 \equiv E_1^2 - M_1^2. \tag{3.167}$$

Next combine Eq.(3.166) with Eq.(3.163) and Eq.(3.164) to yield finally our four-component equation

$$[E_1 D_1^{-+}\frac{1}{\mathcal{D}}(M_2 D_1^{++} - M_1 D_2^{++}) + M_1 D_1^{--}\frac{1}{\mathcal{D}}(E_2 D_1^{++} + E_1 D_2^{++})]\phi_+ = \mathcal{B}^2\phi_+ \tag{3.168}$$

Due to the relativistic “ third law ”, the above equation is symmetrical in particle one and particle two. Using

$$\epsilon_1^2 - \epsilon_2^2 = m_1^2 - m_2^2, \tag{3.169}$$

$$or \quad \epsilon_1^2 - m_1^2 = \epsilon_2^2 - m_2^2 = b^2(w). \tag{3.170}$$

we find that

$$\mathcal{B}^2 = {E_1}^2 - {M_1}^2 = {E_2}^2 - {M_2}^2$$

$$= b^2(w) + (\epsilon_1^2 + \epsilon_2^2)\text{sh}^2(J) + 2\epsilon_1\epsilon_2\text{sh}(J)\text{ch}(J) - (m_1^2 + m_2^2)\text{sh}^2(L) - 2m_1m_2\text{sh}(L)\text{ch}(L). \tag{3.171}$$

By the definitions of $D_i^{\pm\pm}$, Eq.(3.168) is a second-order Schrödinger-like eigenvalue equation for the newly defined wavefunction ϕ_+ in the form.

$$(p_\perp^2 + \Phi(r, \sigma_1, \sigma_2, w))\phi_+ = b^2(w)\phi_+. \tag{3.172}$$

Eq(3.171) provide us with the primary spin independent part of Φ, the quasipotential. Note that in the c.m. system $p_\perp^2 = \mathbf{p}^2$.

We now proceed to review how the form for Eqs.(3.168) displays all the general spin dependent structures in $\Phi(r, \sigma_1, \sigma_2, w)$ explicitly. Very similar to what appears in nonrelativistic formalism such as seen in the Hamada-Johnson and Yale group models(as well as the nonrelativistic limit of Gross's equation). Our new contribution begin from Eq.(3.173). We proceed to the case of L, J, $\mathcal{G}$, C interactions acting simultaneously. We express Eq.(3.168) explicitly in terms of its matrix (σ_1, σ_2), and operator p structure in the c.m. system ($\hat{P} = (1, \mathbf{0})$).

We use the following abbreviations for the kinetic portions by defining the scalar h

and vectors $\mathbf{k}, \mathbf{z}, \mathbf{d}, \mathbf{o}$, functions:

$$\begin{aligned}
h &\equiv \mathrm{e}^{\mathcal{G}}, \\
\mathbf{k} &\equiv \frac{1}{2}\nabla \ln(h), \\
\mathbf{z} &\equiv \frac{1}{2}\nabla(-C + J - L) \\
\mathbf{d} &\equiv \frac{1}{2}\nabla(C + J + L) \\
\mathbf{o} &\equiv \frac{1}{2}\nabla(C - J - L).
\end{aligned}$$

We are working in the c.m. frame (i.e. $x_\perp = (\mathbf{r}, 0)$), so all the interaction functions $(L(x_\perp), J(x_\perp), C(x_\perp), \mathcal{G}(x_\perp))$ are functions of r, $F = F(r)$. Using

$$\nabla F(r) = \frac{dF(r)}{dr}\hat{\mathbf{r}} \equiv F'(r)\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ denotes the unit vector, we can express the vector terms $(\mathbf{k}, \mathbf{z}, \mathbf{d}, \mathbf{o})$ explicitly in terms of the unit vector $\hat{\mathbf{r}}$. With these newly defined terms the first-order differential operators of Eq.(3.159) to Eq.(3.162) become

$$D_1^{++} \equiv h[\sigma_1 \cdot \mathbf{p} - i\sigma_2 \cdot (\mathbf{z} + \mathbf{k}\sigma_1 \cdot \sigma_2)],$$

$$D_2^{++} \equiv h[\sigma_2 \cdot \mathbf{p} - i\sigma_1 \cdot (\mathbf{z} + \mathbf{k}\sigma_1 \cdot \sigma_2)],$$

$$D_1^{-+} \equiv h[\sigma_1 \cdot \mathbf{p} - i\sigma_2 \cdot (\mathbf{d} + \mathbf{k}\sigma_1 \cdot \sigma_2)],$$

$$D_1^{--} \equiv h[\sigma_1 \cdot \mathbf{p} - i\sigma_2 \cdot (\mathbf{o} + \mathbf{k}\sigma_1 \cdot \sigma_2)].$$

Long and Crater define the following terms in Eq.(3.168).

$$F_1 \equiv \frac{M_2}{\mathcal{D}}$$

$$F_2 \equiv \frac{M_1}{\mathcal{D}}$$

$$F_3 \equiv \frac{E_2}{\mathcal{D}}$$

$$F_4 \equiv \frac{E_1}{\mathcal{D}}$$

Using these definitions, Eq.(3.168) becomes

$$h[E_1[\sigma_1 \cdot \mathbf{p} - i\sigma_2 \cdot (\mathbf{d} + \mathbf{k}\sigma_1 \cdot \sigma_2)]hF_1[\sigma_1 \cdot \mathbf{p} - i\sigma_2 \cdot (\mathbf{z} + \mathbf{k}\sigma_1 \cdot \sigma_2)]\phi_+ \qquad (a)$$

$$+h[M_1[\sigma_1 \cdot \mathbf{p} - i\sigma_2 \cdot (\mathbf{o} + \mathbf{k}\sigma_1 \cdot \sigma_2)]hF_3[\sigma_1 \cdot \mathbf{p} - i\sigma_2 \cdot (\mathbf{z} + \mathbf{k}\sigma_1 \cdot \sigma_2)]\phi_+ \qquad (b)$$

$$-h[E_1[\sigma_1\cdot\mathbf{p}-i\sigma_2\cdot(\mathbf{d}+\mathbf{k}\sigma_1\cdot\sigma_2)]hF_2[\sigma_2\cdot\mathbf{p}-i\sigma_1\cdot(\mathbf{z}+\mathbf{k}\sigma_1\cdot\sigma_2)]\phi_+ \qquad (c)$$

$$+h[M_1[\sigma_1\cdot\mathbf{p}-i\sigma_2\cdot(\mathbf{o}+\mathbf{k}\sigma_1\cdot\sigma_2)]hF_4[\sigma_2\cdot\mathbf{p}-i\sigma_1\cdot(\mathbf{z}+\mathbf{k}\sigma_1\cdot\sigma_2)]\phi_+ \qquad (d)$$

$$=\mathcal{B}^2\phi_+. \tag{3.173}$$

For future reference we will refer to the four sets of terms on the left hand side as the Eq.(3.173) (a),(b),(c),(d) term.

Now we proceed in a different derivation than Long and Crater's derivation[31]. The aim is to produce a Schrödinger like form involving the Pauli matrices for both particles.

Substitute $\mathbf{d}$, h, F_1, $\mathbf{z}$, $\mathbf{k}$'s expressions to (a) term of Eq.(3.173), we obtain

$$(a)\ \ term=\mathrm{e}^{\mathcal{G}}E_1\{[\sigma_1\cdot\mathbf{p}-\frac{i}{2}\sigma_2\cdot\nabla(C+J+L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]$$

$$\times\mathrm{e}^{\mathcal{G}}\frac{M_2}{\mathcal{D}}[\sigma_1\cdot\mathbf{p}-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]\} \tag{3.174}$$

working out the commutation relation of $\sigma_1\cdot\mathbf{p}$ in above expression, we can find the (a) term is(see Appendix B)

(a) term= $\mathrm{e}^{\mathcal{G}}E_1\times$

$$\{\mathrm{e}^{\mathcal{G}}\frac{M_2}{\mathcal{D}}[\mathbf{p}^2-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)(\sigma_1\cdot\mathbf{p})-\frac{i}{2}\nabla\mathcal{G}\cdot[(\mathbf{p}+i(\sigma_1\times\mathbf{p})-(\sigma_1\cdot\sigma_2)\mathbf{p}+\sigma_1(\sigma_2\cdot\mathbf{p})-i(\sigma_2\times\mathbf{p})]]$$

$$+\frac{1}{i}\sigma_1\cdot\partial[\mathrm{e}^{\mathcal{G}}\frac{M_2}{\mathcal{D}}[\sigma_1\cdot\mathbf{p}-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]]-$$

$$\frac{i}{2}[\sigma_2\cdot\nabla(C+J+L)+\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]\mathrm{e}^{\mathcal{G}}\frac{M_2}{\mathcal{D}}[\sigma_1\cdot\mathbf{p}-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]\}$$

Likewise we can find (b),(c),(d) term.

(b) term= $e^{\mathcal{G}} M_1 \times$

$$\{e^{\mathcal{G}}\frac{E_2}{\mathcal{D}}[\mathbf{p}^2-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)(\sigma_1\cdot\mathbf{p})-\frac{i}{2}\nabla\mathcal{G}\cdot[(\mathbf{p}+i(\sigma_1\times\mathbf{p})-(\sigma_1\cdot\sigma_2)\mathbf{p}+\sigma_1(\sigma_2\cdot\mathbf{p})-i(\sigma_2\times\mathbf{p})]]$$

$$+\frac{1}{i}\sigma_1\cdot\partial[e^{\mathcal{G}}\frac{E_2}{\mathcal{D}}[\sigma_1\cdot\mathbf{p}-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]]-$$

$$\frac{i}{2}[\sigma_2\cdot\nabla(C-J-L)+\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]e^{\mathcal{G}}\frac{E_2}{\mathcal{D}}[\sigma_1\cdot\mathbf{p}-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]\}$$

(c) term= $-e^{\mathcal{G}} E_1 \times$

$$\{e^{\mathcal{G}}\frac{M_1}{\mathcal{D}}[(\sigma_2\cdot\mathbf{p})(\sigma_1\cdot\mathbf{p})-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)(\sigma_1\cdot\mathbf{p})-\frac{i}{2}\nabla\mathcal{G}\cdot[(\sigma_2(\sigma_1\cdot\mathbf{p})-(\sigma_1\cdot\sigma_2)\mathbf{p}+\sigma_1(\sigma_2\cdot\mathbf{p})$$

$$+i(\sigma_2\times\mathbf{p})]]+\frac{1}{i}\sigma_1\cdot\partial[e^{\mathcal{G}}\frac{M_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_2+i\sigma_2\times\sigma_1)]]-$$

$$\frac{i}{2}[\sigma_2\cdot\nabla(C+J+L)+\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]e^{\mathcal{G}}\frac{M_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_2+i\sigma_2\times\sigma_1)]\}$$

(d) term= $e^{\mathcal{G}} M_1 \times$

$$\{e^{\mathcal{G}}\frac{E_1}{\mathcal{D}}[(\sigma_2\cdot\mathbf{p})(\sigma_1\cdot\mathbf{p})-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)(\sigma_1\cdot\mathbf{p})-\frac{i}{2}\nabla\mathcal{G}\cdot[(\sigma_2(\sigma_1\cdot\mathbf{p})-(\sigma_1\cdot\sigma_2)\mathbf{p}+\sigma_1(\sigma_2\cdot\mathbf{p})$$

$$+i(\sigma_2\times\mathbf{p})]]+\frac{1}{i}\sigma_1\cdot\partial[e^{\mathcal{G}}\frac{E_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_2+i\sigma_2\times\sigma_1)]]-$$

$$\frac{i}{2}[\sigma_2\cdot\nabla(C-J-L)+\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]e^{\mathcal{G}}\frac{E_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_2+i\sigma_2\times\sigma_1)]\}$$

Combining all the (a),(b),(c),(d) term, we get(see Appendix B)

(a) + (b) + (c) + (d) term=

$$e^{\mathcal{G}}\{e^{\mathcal{G}}[\mathbf{p}^2-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)(\sigma_1\cdot\mathbf{p})-\frac{i}{2}\nabla\mathcal{G}\cdot(\mathbf{p}+i(\sigma_1\times\mathbf{p})-(\sigma_1\cdot\sigma_2)\mathbf{p}+\sigma_1(\sigma_2\cdot\mathbf{p})-i(\sigma_2\times\mathbf{p}))]$$

$$+\frac{E_1}{i}\sigma_1\cdot\nabla[e^{\mathcal{G}}\frac{M_2}{\mathcal{D}}[\sigma_1\cdot\mathbf{p}-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]]$$

$$+\frac{M_1}{i}\sigma_1\cdot\nabla[e^{\mathcal{G}}\frac{E_2}{\mathcal{D}}[\sigma_1\cdot\mathbf{p}-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]]$$

$$-\frac{iE_1}{2}[\sigma_2 \cdot \nabla(C+J+L) + \nabla\mathcal{G} \cdot (\sigma_1 + i\sigma_1 \times \sigma_2)] \times$$

$$\times \mathrm{e}^{\mathcal{G}} \frac{M_2}{\mathcal{D}}[\sigma_1 \cdot \mathbf{p} - \frac{i}{2}\sigma_2 \cdot \nabla(-C+J-L) - \frac{i}{2}\nabla\mathcal{G} \cdot (\sigma_1 + i\sigma_1 \times \sigma_2)]$$

$$-\frac{iM_1}{2}[\sigma_2 \cdot \nabla(C-J-L) + \nabla\mathcal{G} \cdot (\sigma_1 + i\sigma_1 \times \sigma_2)] \times$$

$$\times \mathrm{e}^{\mathcal{G}} \frac{E_2}{\mathcal{D}}[\sigma_1 \cdot \mathbf{p} - \frac{i}{2}\sigma_2 \cdot \nabla(-C+J-L) - \frac{i}{2}\nabla\mathcal{G} \cdot (\sigma_1 + i\sigma_1 \times \sigma_2)]\}$$

$$+\mathrm{e}^{\mathcal{G}}\{\frac{M_1}{i}\sigma_1 \cdot \nabla[\mathrm{e}^{\mathcal{G}} \frac{E_1}{\mathcal{D}}[\sigma_2 \cdot \mathbf{p} - \frac{i}{2}\sigma_1 \cdot \nabla(-C+J-L) - \frac{i}{2}\nabla\mathcal{G} \cdot (\sigma_2 + i\sigma_2 \times \sigma_1)]]$$

$$-\frac{E_1}{i}\sigma_1 \cdot \nabla[\mathrm{e}^{\mathcal{G}} \frac{M_1}{\mathcal{D}}[\sigma_2 \cdot \mathbf{p} - \frac{i}{2}\sigma_1 \cdot \nabla(-C+J-L) - \frac{i}{2}\nabla\mathcal{G} \cdot (\sigma_2 + i\sigma_2 \times \sigma_1)]]$$

$$+i\sigma_2 \cdot \nabla(J+L)[\mathrm{e}^{\mathcal{G}} \frac{M_1 E_1}{\mathcal{D}}[\sigma_2 \cdot \mathbf{p} - \frac{i}{2}\sigma_1 \cdot \nabla(-C+J-L) - \frac{i}{2}\nabla\mathcal{G} \cdot (\sigma_2 + i\sigma_2 \times \sigma_1)]]\}$$

We simplify the above expression by using identities involving σ_1 and σ_2 and after a lengthy derivation, we can group above equations by the $\mathbf{p}^2$ term , Darwin term $(\hat{\mathbf{r}}\cdot\mathbf{p})$, spin-orbit angular momentum term $L\cdot(\sigma_1+\sigma_2)$, spin-orbit angular momentum difference term $L\cdot(\sigma_1-\sigma_2)$, spin-spin term $(\sigma_1\cdot\sigma_2)$, tensor term $(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}})$, additional spin dependent terms $L\cdot(\sigma_1\times\sigma_2)$ and $(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\mathbf{p})+(\sigma_2\cdot\hat{\mathbf{r}})(\sigma_1\cdot\mathbf{p})$ and spin independent terms. The final result for above expression is(see Appendix B)

(a) + (b) + (c) + (d) term=

$$\mathrm{e}^{2\mathcal{G}}\{\mathbf{p}^2 - i[2\mathcal{G}' - \frac{E_2M_2+M_1E_1}{\mathcal{D}}(J+L)']\hat{\mathbf{r}}\cdot\mathbf{p} - \frac{1}{2}\nabla^2\mathcal{G} - \frac{1}{4}\mathcal{G}'^2$$

$$-\frac{1}{4}(C+J-L)'(-C+J-L)' + \frac{1}{2}\frac{E_2M_2+M_1E_1}{\mathcal{D}}\mathcal{G}'(J+L)'$$

$$+\frac{L\cdot(\sigma_1+\sigma_2)}{r}[\mathcal{G}' - \frac{1}{2}\frac{E_2M_2+M_1E_1}{\mathcal{D}}(J+L)'] - \frac{L\cdot(\sigma_1-\sigma_2)}{r}\frac{1}{2}\frac{E_2M_2-M_1E_1}{\mathcal{D}}(J+L)'$$

$$+(\sigma_1\cdot\sigma_2)[\frac{1}{2}\nabla^2\mathcal{G} + \frac{1}{2}\mathcal{G}'^2 - \frac{1}{2}\frac{E_2M_2+M_1E_1}{\mathcal{D}}\mathcal{G}'(J+L)' - \frac{1}{2}\mathcal{G}'C' - \frac{1}{2}\frac{\mathcal{G}'}{r} - \frac{1}{2}\frac{(-C+J-L)'}{r}]$$

$$+(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}})[-\frac{1}{2}\nabla^2(-C+J-L) - \frac{1}{2}\nabla^2\mathcal{G} - \mathcal{G}'(-C+J-L)' - \mathcal{G}'^2 + \frac{3}{2r}\mathcal{G}'$$

$$+\frac{3}{2r}(-C+J-L)'+\frac{1}{2}\frac{E_2M_2+M_1E_1}{\mathcal{D}}(J+L)'(\mathcal{G}-C+J-L)']$$

$$+\frac{L\cdot(\sigma_1\times\sigma_2)}{r}\frac{i}{2}\frac{M_2E_1-M_1E_2}{\mathcal{D}}(J+L)'-((\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\mathbf{p})+(\sigma_2\cdot\hat{\mathbf{r}})(\sigma_1\cdot\mathbf{p}))\frac{i(J-L)'}{2}\}$$

So our Eq.(3.173) becomes

$$\mathrm{e}^{2\mathcal{G}}\{\mathbf{p}^2-i[2\mathcal{G}'-\frac{E_2M_2+M_1E_1}{\mathcal{D}}(J+L)']\hat{\mathbf{r}}\cdot\mathbf{p}-\frac{1}{2}\nabla^2\mathcal{G}-\frac{1}{4}\mathcal{G}'^2$$

$$-\frac{1}{4}(C+J-L)'(-C+J-L)'+\frac{1}{2}\frac{E_2M_2+M_1E_1}{\mathcal{D}}\mathcal{G}'(J+L)'$$

$$+\frac{L\cdot(\sigma_1+\sigma_2)}{r}[\mathcal{G}'-\frac{1}{2}\frac{E_2M_2+M_1E_1}{\mathcal{D}}(J+L)']-\frac{L\cdot(\sigma_1-\sigma_2)}{r}\frac{1}{2}\frac{E_2M_2-M_1E_1}{\mathcal{D}}(J+L)'$$

$$+(\sigma_1\cdot\sigma_2)[\frac{1}{2}\nabla^2\mathcal{G}+\frac{1}{2}\mathcal{G}'^2-\frac{1}{2}\frac{E_2M_2+M_1E_1}{\mathcal{D}}\mathcal{G}'(J+L)'-\frac{1}{2}\mathcal{G}'C'-\frac{1}{2}\frac{\mathcal{G}'}{r}-\frac{1}{2}\frac{(-C+J-L)'}{r}]$$

$$+(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}})[-\frac{1}{2}\nabla^2(-C+J-L)-\frac{1}{2}\nabla^2\mathcal{G}-\mathcal{G}'(-C+J-L)'-\mathcal{G}'^2+\frac{3}{2r}\mathcal{G}'$$

$$+\frac{3}{2r}(-C+J-L)'+\frac{1}{2}\frac{E_2M_2+M_1E_1}{\mathcal{D}}(J+L)'(\mathcal{G}-C+J-L)']$$

$$+\frac{L\cdot(\sigma_1\times\sigma_2)}{r}\frac{i}{2}\frac{M_2E_1-M_1E_2}{\mathcal{D}}(J+L)'-((\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\mathbf{p})+(\sigma_2\cdot\hat{\mathbf{r}})(\sigma_1\cdot\mathbf{p}))\frac{i(J-L)'}{2}\}\phi_+$$

$$=\mathcal{B}^2\phi_+. \tag{3.175}$$

3.3 The Radial Eigenvalue Equations

For singlet states $^1S_0$, $^1P_1$, $^1D_2$ and triplet states $^3P_0$, $^3P_1$, $^3S_1$, $^3D_1$, we can get their radial eigenvalue equations from Eq.(3.175) for equal mass case as following(see Appendix B)

$s=0, j=l$

$$\{-\frac{d^2}{dr^2}+\frac{j(j+1)}{r^2}-(2\mathcal{G}-\ln(\mathcal{D})-J+L)'(\frac{d}{dr}-\frac{1}{r})$$

$$+\frac{1}{2}\nabla^2(-C+J-L-3\mathcal{G})-\frac{1}{4}(C+J-L-\mathcal{G}+2\ln(\mathcal{D}))'(-C+J-L-3\mathcal{G})'\}u_{j0j}$$

$$=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}u_{j0j},$$

$s = 1, j = l$

$$\{-\frac{d^2}{dr^2} + \frac{j(j+1)}{r^2} - (2\mathcal{G} + J - L - \ln(\mathcal{D}))'\frac{d}{dr} + \frac{(-C + J - L + \mathcal{G})'}{r}$$

$$-\frac{1}{2}\nabla^2(-C + J - L + \mathcal{G}) + \frac{1}{4}(2\ln(\mathcal{D}) - (C + J - L + 3\mathcal{G}))'(-C + J - L + \mathcal{G})'\}u_{j1j}$$

$$= \mathcal{B}^2 \mathrm{e}^{-2\mathcal{G}} u_{j1j}.$$

The above two equations are uncoupled because we neglect the neutron and proton mass difference. Below are the coupled equations for equal mass due to the tensor term

$s = 1, j = l + 1$

$$\{(-\frac{d^2}{dr^2} + \frac{j(j-1)}{r^2}) + [\ln(\mathcal{D}) - 2\mathcal{G} - \frac{1}{2j+1}(J - L)]'\frac{d}{dr}$$

$$[-j\ln(\mathcal{D}) + \frac{1}{2j+1}((4j^2 + j + 1)\mathcal{G} + J - L + (j-1)C)]'\frac{1}{r} - \frac{1}{2}\mathcal{G}'C' + \frac{1}{4}(C'^2 - (J-L)'^2)$$

$$+\frac{1}{2j+1}((-\frac{1}{2}\nabla^2(J-L+\mathcal{G}-C)+\mathcal{G}'(\frac{2j-3}{4}\mathcal{G}-J+L+C)'+\frac{1}{2}\ln'(\mathcal{D})(\mathcal{G}+J-L-C)')\}u_-$$

$$+\frac{\sqrt{j(j+1)}}{2j+1}\{-2[J-L]'\frac{d}{dr} + [(J-L)(1-2j) + 3\mathcal{G} - 3C]'\frac{1}{r}$$

$$-\nabla^2(J-L+\mathcal{G}-C)+(J-L+\mathcal{G}-C)'(\ln(\mathcal{D})-2\mathcal{G})'\}u_+=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}u_-,$$

$s=1, j=l-1$

$$\{(-\frac{d^2}{dr^2}+\frac{(j+1)(j+2)}{r^2})+[\ln(\mathcal{D})-2\mathcal{G}+\frac{1}{2j+1}(J-L)]'\frac{d}{dr}$$

$$[(j+1)\ln(\mathcal{D})-\frac{1}{2j+1}((4j^2+7j+4)\mathcal{G}+J-L-(j+2)C)]'\frac{1}{r}-\frac{1}{2}\mathcal{G}'C'+\frac{1}{4}(C'^2-(J-L)'^2)$$

$$+\frac{1}{2j+1}((\frac{1}{2}\nabla^2(J-L+\mathcal{G}-C)+\mathcal{G}'(\frac{2j+5}{4}\mathcal{G}+J-L-C)'-\frac{1}{2}\ln'(\mathcal{D})(\mathcal{G}+J-L-C)')\}u_+$$

$$+\frac{\sqrt{j(j+1)}}{2j+1}\{-2[J-L]'\frac{d}{dr}+[(J-L)(2j+3)+3\mathcal{G}-3C]'\frac{1}{r}$$

$$-\nabla^2(J-L+\mathcal{G}-C)+(J-L+\mathcal{G}-C)'(\ln(\mathcal{D})-2\mathcal{G})'\}u_-=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}u_+.$$

The above radial eigenvalue equations agree with Long's results[31]; we obtain the same equations by using a different derivation method. All of above equations have a first derivative term, this is a difference with the standard Schrödinger like equation. Before we can apply the techniques which have been already developed for the Schrödinger-like system in nonrelativistic quantum mechanics, we must get rid of these first derivative terms . Without the first derivative terms, we can use the phase shift equations which we will discuss in the next chapter. For the uncoupled states, it is pretty straightforward. For the coupled states, the radial eigenvalue equations is in a matrix form and the process in getting rid of the first derivative term is very complicated. We were not successful in getting rid of first derivative terms for coupled states starting with the radial eigenvalue equations. However, we will describe later a different approach that is very successful.

3.4 Removal Of The First Derivative Terms

First, let us start from the uncoupled states, we take ${}^1S_0$ state as an example, the radial eigenvalue equation for singlet ${}^1S_0$ state is

$$\{-\frac{d^2}{dr^2} - (2\mathcal{G} - \ln(\mathcal{D}) - J + L)'(\frac{d}{dr} - \frac{1}{r}) + \frac{1}{2}\nabla^2(-C + J - L - 3\mathcal{G})$$

$$-\frac{1}{4}(C + J - L - \mathcal{G} + 2\ln(\mathcal{D}))'(-C + J - L - 3\mathcal{G})'\}u = \mathcal{B}^2 \mathrm{e}^{-2\mathcal{G}} u, \tag{3.176}$$

Let $u = fv$, substitute to above equation and let $-2f'v' - K'fv' = 0$, we get

$$f = f_0 e^{-\frac{K}{2}},$$

$$f' = -f_0(\frac{K'}{2})e^{-\frac{K}{2}},$$

$$f'' = f_0(\frac{K'}{2})^2 e^{-\frac{K}{2}} - f_0(\frac{K''}{2}) e^{-\frac{K}{2}},$$

$$K = 2\mathcal{G} - \ln(\mathcal{D}) - J + L \tag{3.177}$$

f_0 is a constant, the above equation become

$$\{-\frac{d^2}{dr^2}+\frac{K^{'2}}{4}+\frac{K''}{2}+\frac{K^{'}}{r}+\frac{1}{2}\nabla^2(-C+J-L-3\mathcal{G})-\frac{1}{4}(C+J-L-\mathcal{G}+2\ln(\mathcal{D}))'(-C+J-L-3\mathcal{G})'\}v$$

$$= \mathcal{B}^2 e^{-2\mathcal{G}} v \tag{3.178}$$

This has the desired Schrödinger-like form similar to Reid's non-relativistic equation. So our potential for this Schrödinger-like equation is

$$\Phi(r) = \frac{K^{'2}}{4}+\frac{K''}{2}+\frac{K^{'}}{r}+\frac{1}{2}\nabla^2(-C+J-L-3\mathcal{G})-\frac{1}{4}(C+J-L-\mathcal{G}+2\ln D)^{'}(-C+J-L-3\mathcal{G})'$$

$$-\mathcal{B}^2 e^{-2\mathcal{G}} + b^2(w) \tag{3.179}$$

For the coupled states, the way to get rid of the first derivative term is very complicated, it is significantly different from the above approach for uncoupled states. However, the new approach gives the same result for uncoupled states.

The general form of the eigenvalue equation given in Eq.(3.175) is:

$$[\mathbf{p}^2 - ig'\hat{\mathbf{r}}\cdot\mathbf{p} + \frac{g'}{2r}\vec{L}\cdot(\sigma_1+\sigma_2) - ih'(\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\mathbf{p} + \sigma_2\cdot\hat{\mathbf{r}}\sigma_1\cdot\mathbf{p})$$

$$+k\sigma_1\cdot\sigma_2 + j\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}} + l\vec{L}\cdot(\sigma_1-\sigma_2) + in\vec{L}\cdot(\sigma_1\times\sigma_2) + m]\Psi$$

$$= \mathcal{B}^2 e^{-2\mathcal{G}}\Psi. \tag{3.180}$$

the m term is the spin independent term. For the equal mass case, two terms drop out(see Eq.(3.175)), the above equation becomes

$$[\mathbf{p}^2 - ig'\hat{\mathbf{r}}\cdot\mathbf{p} + \frac{g'}{2r}\vec{L}\cdot(\sigma_1+\sigma_2) - ih'(\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\mathbf{p} + \sigma_2\cdot\hat{\mathbf{r}}\sigma_1\cdot\mathbf{p})$$

$$+k\sigma_1\cdot\sigma_2 + j\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}} + m]\Psi = \mathcal{B}^2 e^{-2\mathcal{G}}\Psi. \tag{3.181}$$

We let

$$\Psi = \exp(F + K\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})\psi \equiv (A + B\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})\psi. \tag{3.182}$$

We find that (see Appendix C)

$$\mathbf{p}\Psi \;=\; (A + B\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})\mathbf{p}\psi - i(A' + B'\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\psi$$

$$-i\frac{B}{r}[(\sigma_1 - \sigma_1 \cdot \hat{\mathbf{r}}\hat{\mathbf{r}})\sigma_2 \cdot \hat{\mathbf{r}} + (\sigma_2 - \sigma_2 \cdot \hat{\mathbf{r}}\hat{\mathbf{r}})\sigma_1 \cdot \hat{\mathbf{r}}]\psi, \tag{3.183}$$

and

$$\frac{g'}{2r}\vec{L} \cdot (\sigma_1 + \sigma_2)\Psi = (A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\frac{g'}{2r}\vec{L} \cdot (\sigma_1 + \sigma_2)\psi$$

$$+\frac{g'}{2r}B[2\sigma_1 \cdot \sigma_2 - 4ir\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p} + 2ir(\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p}) - 6\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}]\psi, \tag{3.184}$$

We thus find that

$$-ig'\hat{\mathbf{r}} \cdot \mathbf{p}\Psi = (A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})(-ig'\hat{\mathbf{r}} \cdot \mathbf{p})\psi + C\psi \tag{3.185}$$

and

$$\begin{aligned} -ih'(\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p})\Psi &= (A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})(-ih'[\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p}])\psi \\ &\quad + D\psi \end{aligned}$$

and finally

$$\begin{aligned} \mathbf{p}^2\Psi &= (A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\mathbf{p}^2\psi - 2i(A' + B'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} \cdot \mathbf{p}\psi \\ &\quad + i\frac{2B}{r}[2\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p} - (\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p}]\psi + E\psi \end{aligned} \tag{3.186}$$

where C and D and E do not involve $\mathbf{p}$ and are given by

$$C = -g'(A' + B'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}), \tag{3.187}$$

$$D = -2h'(\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}A' + B') - 2h'\frac{B}{r}[\vec{L} \cdot (\sigma_1 + \sigma_2) + 2 - \sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}} + \sigma_1 \cdot \sigma_2], \tag{3.188}$$

and

$$E = -(A'' + B''\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}) - \frac{2}{r}(A' + B'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}) - 2\frac{B}{r^2}(\sigma_1 \cdot \sigma_2 - 3\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}). \tag{3.189}$$

The general form of the eigenvalue equation becomes(for detail, see Appendix C)

$$\begin{aligned}
& (A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})[\mathbf{p}^2 - ig'\hat{\mathbf{r}} \cdot \mathbf{p} + \frac{g'}{2r}\vec{L} \cdot (\sigma_1 + \sigma_2) - ih'(\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p})]\psi \\
& +(\frac{g'}{2r}B[2\sigma_1 \cdot \sigma_2 - 4ir\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p} + 2ir(\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p}) - 6\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}] \\
& -2i(A' + B'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} \cdot \mathbf{p} + i\frac{2B}{r}[2\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p} - (\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p})] \\
& +(k\sigma_1 \cdot \sigma_2 + j\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}) + R + m)\psi \\
= \; & \mathcal{B}^2 e^{-2\mathcal{G}}(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\psi
\end{aligned} \tag{3.190}$$

in which $R = C + D + E$. Now, to bring this equation to the desired Schrödinger-like form we multiply both sides by

$$(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})^{-1} = \frac{(A - B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})}{A^2 - B^2} \tag{3.191}$$

and find using the exponential form above that appears in Eq.(3.182), (see Eq.(C.9) to Eq.(C.10) in Appendix C)

$$\begin{aligned}&(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})^{-1}[-2i(A' + B'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})]\hat{\mathbf{r}} \cdot \mathbf{p} \\ &= \quad -2i(F' + K'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} \cdot \mathbf{p},\end{aligned} \tag{3.192}$$

$$\begin{aligned}&(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})^{-1} i\frac{2B}{r}[2\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p} - (\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p})] \\ &= \quad \frac{2i\sinh(K)\cosh(K)}{r}[2\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p} - (\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p})] + G\end{aligned} \tag{3.193}$$

where(see Eq(C.11) in Appendix C)

$$G = -\frac{2\sinh^2(K)}{r^2}\vec{L} \cdot (\sigma_1 + \sigma_2),$$

and

$$(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})^{-1}\frac{g'}{2r}B[2\sigma_1 \cdot \sigma_2 - 4ir\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p}$$

$$+2ir(\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p}) - 6\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}]$$

$$= \frac{ig'\sinh(K)\cosh(K)}{2r}[-4r\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p} + 2r(\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p})$$

$$-2i\sigma_1 \cdot \sigma_2 + 6i\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}] + H \tag{3.194}$$

where

$$H = \frac{g'\sinh^2(K)}{2r}[2\vec{L} \cdot (\sigma_1 + \sigma_2) - 2\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}} + 2\sigma_1 \cdot \sigma_2 + 4].$$

For detail see Eq(C.13) in Appendix C, note G and H do not contain linear $\mathbf{p}$ type of terms. Now collect the three different linear $\mathbf{p}$ type of terms in equation(3.190):

$$(-2iF' - ig')\hat{\mathbf{r}} \cdot \mathbf{p}, \tag{3.195}$$

$$(-2i\frac{\sinh(K)\cosh(K)}{r} - ih' + ig'\sinh(K)\cosh(K))(\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p}), \tag{3.196}$$

$$(4i\frac{\sinh(K)\cosh(K)}{r} - 2i\sinh(K)\cosh(K)g' - 2iK')\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p}. \tag{3.197}$$

If we set the first equation to 0, we obtain the expected result

$$F' = -g'/2. \tag{3.198}$$

If we set $h' = -K'$ and use $\mathbf{p} = \hat{\mathbf{r}}(\hat{\mathbf{r}}.\mathbf{p}) - \frac{\hat{\mathbf{r}}\times L}{r}$ to combine the two expressions(3.196 and 3.197), we get

$$(2\frac{\sinh(K)\cosh(K)}{r} + h' - g'\sinh(K)\cosh(K))\frac{\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}\vec{L}\cdot(\sigma_1+\sigma_2)}{r} \tag{3.199}$$

which contains no $\hat{\mathbf{r}}\cdot\mathbf{p}$. Thus the matrix scale change

$$\Psi = \exp(-g/2)\exp(-h\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})\psi \tag{3.200}$$

eliminates the linear $\mathbf{p}$ terms.

Further note that

$$\begin{aligned} & (A + B\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})^{-1}(k\sigma_1\cdot\sigma_2 + j\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})(A + B\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}) \quad (3.201) \\ = \quad & (k\sigma_1\cdot\sigma_2 + j\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}), \end{aligned}$$

$$(A + B\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})^{-1}C\psi = -g'(F' + K'\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})\psi, \tag{3.202}$$

and(see detail in Appendix C)

$$(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})^{-1} D\psi = -2h'(K' + F'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\psi,$$

$$-2h'\frac{\cosh(K)\sinh(K)}{r}[\vec{L} \cdot (\sigma_1 + \sigma_2) + 2 - \sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}} + \sigma_1 \cdot \sigma_2]\psi$$

$$+2h'\frac{\sinh^2(K)}{r}[\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\vec{L} \cdot (\sigma_1 + \sigma_2) + 3\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}} - \sigma_1 \cdot \sigma_2]\psi. \tag{3.203}$$

also

$$(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})^{-1} E\psi = -[F'' + F'^2 + K'^2 + (2F'K' + K'')\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}]$$

$$-\frac{2}{r}[F' + K'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}] - 2\frac{\cosh(K)\sinh(K)}{r^2}(\sigma_1 \cdot \sigma_2 - 3\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})$$

$$+2\frac{\sinh^2(K)}{r^2}(\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}} - \sigma_1 \cdot \sigma_2 - 2) \tag{3.204}$$

So combining all terms, we have our Schrödinger-like equation

$$\{\mathbf{p}^2 + \frac{g'}{2r}\vec{L} \cdot (\sigma_1 + \sigma_2) + k\sigma_1 \cdot \sigma_2 + j\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}$$

$$+(2\frac{\sinh(K)\cosh(K)}{r}+h'-g'\sinh(K)\cosh(K))\frac{\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}\vec{L}\cdot(\sigma_1+\sigma_2)}{r}$$

$$+\frac{g'\cosh(K)\sinh(K)}{r}(\sigma_1\cdot\sigma-3\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})$$

$$+\frac{g'\sinh^2(K)}{2r}[2\vec{L}\cdot(\sigma_1+\sigma_2)-2\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}+2\sigma_1\cdot\sigma_2+4]$$

$$-2\frac{\sinh^2(K)}{r^2}\vec{L}\cdot(\sigma_1+\sigma_2)-g'(F'+K'\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})-2h'(K'+F'\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})$$

$$-2h'\frac{\cosh(K)\sinh(K)}{r}[\vec{L}\cdot(\sigma_1+\sigma_2)+2-\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}+\sigma_1\cdot\sigma_2]$$

$$+2h'\frac{\sinh^2(K)}{r}[\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}\vec{L}\cdot(\sigma_1+\sigma_2)+3\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}-\sigma_1\cdot\sigma_2]$$

$$-[F''+F'^2+K'^2+(2F'K'+K'')\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}]-\frac{2}{r}[F'+K'\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}]$$

$$-2\frac{\cosh(K)\sinh(K)}{r^2}(\sigma_1\cdot\sigma_2-3\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})$$

$$+2\frac{\sinh^2(K)}{r^2}(\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}-\sigma_1\cdot\sigma_2-2)+m\}\psi$$

$$=\ \mathcal{B}^2e^{-2\mathcal{G}}\psi \tag{3.205}$$

Grouping the above equation by $\mathbf{p}^2$ term , spin independent terms, spin-orbit angular momentum term $L\cdot(\sigma_1+\sigma_2)$, spin-spin term $(\sigma_1\cdot\sigma_2)$, tensor term $(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}})$, additional spin independent term, the above equation becomes

$$
\begin{aligned}
& \{\mathbf{p}^2 + \frac{2g'\sinh^2(K)}{r} - g'F' - 2h'K' - 4h'\frac{\cosh(K)\sinh(K)}{r} \\
& -F'' - F'^2 - K'^2 - \frac{2}{r}F' - 4\frac{\sinh^2(K)}{r^2} \\
& +\vec{L}\cdot(\sigma_1+\sigma_2)[\frac{g'}{2r} + \frac{g'\sinh^2(K)}{r} - \frac{2\sinh^2(K)}{r^2} - 2h'\frac{\cosh(K)\sinh(K)}{r}] \\
+ \quad & \sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}\vec{L}\cdot(\sigma_1+\sigma_2)(2h'\frac{\sinh^2(K)}{r} + 2\frac{\sinh(K)\cosh(K)}{r^2} + \frac{h'}{r} - \frac{g'\sinh(K)\cosh(K)}{r}) \\
+ \quad & \sigma_1\cdot\sigma_2[k + \frac{g'\cosh(K)\sinh(K)}{r} + \frac{g'\sinh^2(K)}{r} - 2h'\frac{\cosh(K)\sinh(K)}{r} \\
& -2h'\frac{\sinh^2(K)}{r} - 2\frac{\cosh(K)\sinh(K)}{r^2} - 2\frac{\sinh^2(K)}{r^2}] + \\
& \sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}[j - \frac{3g'\cosh(K)\sinh(K)}{r} - \frac{g'\sinh^2 K}{r} - g'K' - 2h'F' + \frac{2h'\cosh K\sinh K}{r} \\
& +6h'\frac{\sinh^2(K)}{r} - (2F'K' + K'') - \frac{2}{r}K' + 6\frac{\cosh(K)\sinh(K)}{r^2} + 2\frac{\sinh^2(K)}{r^2}] + m\}\psi
\end{aligned}
$$

$$= \mathcal{B}^2 e^{-2\mathcal{G}}\psi \tag{3.206}$$

The above equation and it's derivation is an important and crucial part of this dissertation. It will provide us with a way to derive phase shift equations using work by other authors who use nonrelativistic methods. First we need the radial form of this equation.

The following are radial eigenvalue equations for singlet states $^1S_0$, $^1P_1$, $^1D_2$ and triplet states $^3P_0$, $^3P_1$, $^3S_1$, $^3D_1$, after getting rid of the first derivative terms(see Appendix B). We emphasis that unlike the potentials used by Reid, Hamada Johnson and the Yale group, our potentials are fixed by the relativistic Dirac equation and we do not have the freedom of choosing different potentials for different angular momentum states.

$^1S_0$, $^1P_1$, $^1D_2$ (a general singlet ^1J_j)

$$\{-\frac{d^2}{dr^2}+\frac{j(j+1)}{r^2}+\frac{(2\mathcal{G}-\ln(\mathcal{D})-J+L)'^2}{4}+\frac{(2\mathcal{G}-\ln(\mathcal{D})-J+L)''}{2}+\frac{(2\mathcal{G}-\ln\mathcal{D}-J+L)'}{r}$$

$$+\frac{1}{2}\nabla^2(-C+J-L-3\mathcal{G})-\frac{1}{4}(C+J-L-\mathcal{G}+2\ln(\mathcal{D}))'(-C+J-L-3\mathcal{G})'\}v=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}v,$$

Our radial eigenvalue equations for singlet states $^1S_0$, $^1P_1$, $^1D_2$ appear to have the same potential forms, except $\frac{j(j+1)}{r^2}$ for the angular momentum terms. Later, we shall show that their potentials actually are different due to the inclusion of isospin $\tau_1 \cdot \tau_2$

terms. Comparing the above equation with Eq.(3.172), we can see our potential for above equation is

$$\Phi(r) = \frac{(2\mathcal{G} - \ln(\mathcal{D}) - J + L)'^2}{4} + \frac{(2\mathcal{G} - \ln(\mathcal{D}) - J + L)''}{2} + \frac{(2\mathcal{G} - \ln(\mathcal{D}) - J + L)'}{r}$$

$$+\frac{1}{2}\nabla^2(-C+J-L-3\mathcal{G}) - \frac{1}{4}(C+J-L-\mathcal{G}+2\ln D)'(-C+J-L-3\mathcal{G})' - \mathcal{B}^2 e^{-2\mathcal{G}} + b^2(w)$$

For the $^3P_0$ state the radial eigenvalue equation is

$$\{-\frac{d^2}{dr^2} + \frac{2}{r^2} + \frac{(2\mathcal{G} - \ln(\mathcal{D}) - J + L)'^2}{4} + \frac{(2\mathcal{G} - \ln(\mathcal{D}) - J + L)''}{2}$$

$$+\frac{(\ln(\mathcal{D}) - (4\mathcal{G} + J - L - 2C))'}{r} + \frac{1}{2}\nabla^2(-C + J - L + \mathcal{G}) - \frac{1}{2}\mathcal{G}'C' + \frac{1}{4}(C'^2 - (J - L)'^2)$$

$$+\mathcal{G}'(\frac{5}{4}\mathcal{G} + J - L - C)' - \frac{1}{2}\ln'(\mathcal{D})(J - L - C + \mathcal{G})'\}v = \mathcal{B}^2 \mathrm{e}^{-2\mathcal{G}} v,$$

so that, our potential for above Schrödinger-like equation in this case is

$$\Phi(r) = \frac{(2\mathcal{G} - \ln(\mathcal{D}) - J + L)'^2}{4} + \frac{(2\mathcal{G} - \ln(\mathcal{D}) - J + L)''}{2} + \frac{(\ln(\mathcal{D}) - (4\mathcal{G} + J - L - 2C))'}{r}$$

$$+\frac{1}{2}\nabla^2(-C + J - L + \mathcal{G}) - \frac{1}{2}\mathcal{G}'C' + \frac{1}{4}(C'^2 - (J - L)'^2) + \mathcal{G}'(\frac{5}{4}\mathcal{G} + J - L - C)'$$

$$-\frac{1}{2}\ln'(\mathcal{D})(J - L - C + \mathcal{G})' - \mathcal{B}^2 \mathrm{e}^{-2\mathcal{G}} + b^2(w),$$

For the $^3P_1$ state the radial eigenvalue equation is

$$\{-\frac{d^2}{dr^2}+\frac{2}{r^2}+\frac{(2\mathcal{G}-\ln(\mathcal{D})+J-L)'^2}{4}+\frac{(2\mathcal{G}-\ln(\mathcal{D})+J-L)''}{2}+\frac{(\mathcal{G}+J-L-C)'}{r}$$

$$-\frac{1}{2}\nabla^2(-C+J-L+\mathcal{G})+\frac{1}{4}(2\ln(\mathcal{D})-(C+J-L+3\mathcal{G}))'(J-L-C+\mathcal{G})'\}v=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}v,$$

so our potential for the above Schrödinger-like equation is(the general form for arbitrary j is given in Appendix C)

$$\Phi(r)=\frac{(2\mathcal{G}-\ln(\mathcal{D})+J-L)'^2}{4}+\frac{(2\mathcal{G}-\ln(\mathcal{D})+J-L)''}{2}+\frac{(\mathcal{G}+J-L-C)'}{r}$$

$$-\frac{1}{2}\nabla^2(-C+J-L+\mathcal{G})+\frac{1}{4}(2\ln(\mathcal{D})-(C+J-L+3\mathcal{G}))'(J-L-C+\mathcal{G})'-\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}+b^2(w),$$

The $^3S_1$ and $^3D_1$are coupled states described by u_- and u_+ and their radial eigenvalue equations are

$$\{-\frac{d^2}{dr^2}+\frac{8}{3}\frac{g'\sinh^2(h)}{r}+\frac{16}{3}\frac{h'\cosh(h)\sinh(h)}{r}-\frac{16}{3}\frac{\sinh^2(h)}{r^2}+\frac{(2\mathcal{G}'-\ln'(\mathcal{D}))^2}{4}+\frac{(J-L)'^2}{4}$$

$$+\frac{(2\mathcal{G}'-\ln'(\mathcal{D}))(J-L)'}{6}+\frac{(2\mathcal{G}''-\ln''(\mathcal{D}))}{2}+\frac{(J-L)''}{6}+\frac{(2\mathcal{G}'-\ln'(\mathcal{D}))}{r}+\frac{(J-L)'}{3r}+$$

$$\frac{1}{3}[-\frac{1}{2}\nabla^2(-C+J-L+\mathcal{G})-\mathcal{G}'(J-L-C+\mathcal{G})'+\frac{1}{2}\ln'(\mathcal{D})(\mathcal{G}+J-L-C)')]$$

$$+\frac{1}{4}\mathcal{G}'^2-\frac{1}{2}\mathcal{G}'C'-\frac{1}{4}(C+J-L)'(-C+J-L)'\}u_-+$$

$$\frac{2\sqrt{2}}{3}\{\frac{3g'\cosh(h)\sinh(h)}{r}-\frac{g'\sinh^2(h)}{r}-\frac{2h'\cosh(h)\sinh(h)}{r}+\frac{6h'\sinh^2(h)}{r}$$

$$-\frac{6\cosh(h)\sinh(h)}{r^2}+\frac{2\sinh^2(h)}{r^2}-\frac{1}{2}\nabla^2(-C+J-L+\mathcal{G})-\mathcal{G}'(J-L-C+\mathcal{G})'$$

$$+\frac{3(\mathcal{G}+J-L-C)'}{2r}+\frac{1}{2}\ln'(\mathcal{D})(\mathcal{G}+J-L-C)'+\frac{(2\mathcal{G}'-\ln'(\mathcal{D}))(J-L)'}{2}+\frac{(J-L)''}{2}$$

$$+\frac{(J-L)'}{r}\}u_+=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}u_-$$

and

$$\{-\frac{d^2}{dr^2}+\frac{6}{r^2}-\frac{8}{3}\frac{g'\sinh^2(h)}{r}-\frac{16}{3}\frac{h'\cosh(h)\sinh(h)}{r}+\frac{16}{3}\frac{\sinh^2(h)}{r^2}$$

$$+\frac{(2\mathcal{G}'-\ln'(\mathcal{D}))^2}{4}+\frac{(J-L)'^2}{4}-\frac{(2\mathcal{G}'-\ln'(\mathcal{D}))(J-L)'}{6}+\frac{(2\mathcal{G}''-\ln''(\mathcal{D}))}{2}$$

$$-\frac{(J-L)''}{6}-\frac{2(2\mathcal{G}'-\ln'(\mathcal{D}))}{r}+\frac{2(J-L)'}{3r}-\frac{(\mathcal{G}+J-L-C)'}{r}$$

$$-\frac{1}{3}[-\frac{1}{2}\nabla^2(-C+J-L+\mathcal{G})-\mathcal{G}'(J-L-C+\mathcal{G})'+\frac{1}{2}\ln'(\mathcal{D})(\mathcal{G}+J-L-C)')]$$

$$\frac{1}{4}\mathcal{G}'^2-\frac{1}{2}\mathcal{G}'C'-\frac{1}{4}(C+J-L)'(-C+J-L)'\}u_+ +$$

$$\{\frac{2\sqrt{2}}{3}[\frac{3g'\cosh(h)\sinh(h)}{r}-\frac{g'\sinh^2(h)}{r}-\frac{2h'\cosh(h)\sinh(h)}{r}+\frac{6h'\sinh^2(h)}{r}$$

$$-\frac{6\cosh(h)\sinh(h)}{r^2}+\frac{2\sinh^2(h)}{r^2}-\frac{1}{2}\nabla^2(-C+J-L+\mathcal{G})-\mathcal{G}'(J-L-C+\mathcal{G})'$$

$$+\frac{3(\mathcal{G}+J-L-C)'}{2r}+\frac{1}{2}\ln'(\mathcal{D})(\mathcal{G}+J-L-C)'+\frac{(2\mathcal{G}'-\ln'(\mathcal{D}))(J-L)'}{2}$$

$$+\frac{(J-L)''}{2}+\frac{(J-L)'}{r}]-4\sqrt{2}[\frac{2h'\sinh^2(h)}{r}-\frac{2\cosh(h)\sinh(h)}{r^2}$$

$$+\frac{h'}{r}+\frac{g'\cosh(h)\sinh(h)}{r}]\}u_- = \mathcal{B}^2 \mathrm{e}^{-2\mathcal{G}} u_+$$

here the $h = \frac{J-L}{2}$ and $g = 2\mathcal{G} - \ln(\mathcal{D})$, and note that g and $\mathcal{G}$ are different variables. The above equations can be put in the form

$$\{-\frac{d^2}{dr^2} + \Phi_{11}(r)\}u_- + \Phi_{12}(r)u_+ = \mathcal{B}^2 \mathrm{e}^{-2\mathcal{G}} u_- \tag{3.207}$$

$$\{-\frac{d^2}{dr^2} + \frac{6}{r^2} + \Phi_{22}(r)\}u_+ + \Phi_{21}(r)u_- = \mathcal{B}^2 \mathrm{e}^{-2\mathcal{G}} u_+ \tag{3.208}$$

where

$$\begin{aligned}\Phi_{11}(r) \quad = \quad & \{\frac{8}{3}\frac{g'\sinh^2(h)}{r} + \frac{16}{3}\frac{h'\cosh(h)\sinh(h)}{r} - \frac{16}{3}\frac{\sinh^2(h)}{r^2} + \frac{(2\mathcal{G}' - \ln'(\mathcal{D}))^2}{4} \\ & + \frac{(J-L)'^2}{4} + \frac{(2\mathcal{G}' - \ln'(\mathcal{D}))(J-L)'}{6} + \frac{(2\mathcal{G}'' - \ln''(\mathcal{D}))}{2} \\ & + \frac{(J-L)''}{6} + \frac{(2\mathcal{G}' - \ln'(\mathcal{D}))}{r} + \frac{(J-L)'}{3r} \\ & \frac{1}{3}[-\frac{1}{2}\nabla^2(-C + J - L + \mathcal{G}) - \mathcal{G}'(J - L - C + \mathcal{G})' + \frac{1}{2}\ln'(\mathcal{D})(\mathcal{G} + J - L - C)')]\end{aligned}$$

$$\frac{1}{4}\mathcal{G}'^2 - \frac{1}{2}\mathcal{G}'C' - \frac{1}{4}(C+J-L)'(-C+J-L)' - \mathcal{B}^2\mathrm{e}^{-2\mathcal{G}} + b^2(w)\}$$

$$\begin{aligned}
\Phi_{12}(r) \quad = \quad & \frac{2\sqrt{2}}{3}\{\frac{3g'\cosh(h)\sinh(h)}{r} - \frac{g'\sinh^2(h)}{r} - \frac{2h'\cosh(h)\sinh(h)}{r} + \frac{6h'\sinh^2(h)}{r} \\
& -\frac{6\cosh(h)\sinh(h)}{r^2} + \frac{2\sinh^2(h)}{r^2} - \frac{1}{2}\nabla^2(-C+J-L+\mathcal{G}) \\
& -\mathcal{G}'(J-L-C+\mathcal{G})' + \frac{3(\mathcal{G}+J-L-C)'}{2r} + \frac{1}{2}\ln'(\mathcal{D})(\mathcal{G}+J-L-C)' \\
& +\frac{(2\mathcal{G}'-\ln'(\mathcal{D}))(J-L)'}{2} + \frac{(J-L)''}{2} + \frac{(J-L)'}{r}\}
\end{aligned}$$

$$\begin{aligned}
\Phi_{22}(r) \quad = \quad & \{-\frac{8}{3}\frac{g'\sinh^2(h)}{r} - \frac{16}{3}\frac{h'\cosh(h)\sinh(h)}{r} + \frac{16}{3}\frac{\sinh^2(h)}{r^2} + \frac{(2\mathcal{G}'-\ln'(\mathcal{D}))^2}{4} \\
& +\frac{(J-L)'^2}{4} - \frac{(2\mathcal{G}'-\ln'(\mathcal{D}))(J-L)'}{6} + \frac{(2\mathcal{G}''-\ln''(\mathcal{D}))}{2} - \frac{(J-L)''}{6} \\
& -\frac{2(2\mathcal{G}'-\ln'(\mathcal{D}))}{r} + \frac{2(J-L)'}{3r} - \frac{(\mathcal{G}+J-L-C)'}{r} \\
& -\frac{1}{3}[-\frac{1}{2}\nabla^2(-C+J-L+\mathcal{G}) - \mathcal{G}'(J-L-C+\mathcal{G})' + \frac{1}{2}\ln'(\mathcal{D})(\mathcal{G}+J-L-C)')]
\end{aligned}$$

$$\frac{1}{4}\mathcal{G}'^2 - \frac{1}{2}\mathcal{G}'C' - \frac{1}{4}(C+J-L)'(-C+J-L)' - \mathcal{B}^2\mathrm{e}^{-2\mathcal{G}} + b^2(w)\}$$

$$\begin{aligned}
\Phi_{21}(r) \quad = \quad & \{\frac{2\sqrt{2}}{3}[\frac{3g'\cosh(h)\sinh(h)}{r} - \frac{g'\sinh^2(h)}{r} - \frac{2h'\cosh(h)\sinh(h)}{r} + \frac{6h'\sinh^2(h)}{r} \\
& -\frac{6\cosh(h)\sinh(h)}{r^2} + \frac{2\sinh^2(h)}{r^2} - \frac{1}{2}\nabla^2(-C+J-L+\mathcal{G}) \\
& -\mathcal{G}'(J-L-C+\mathcal{G})' + \frac{3(\mathcal{G}+J-L-C)'}{2r} + \frac{1}{2}\ln'(\mathcal{D})(\mathcal{G}+J-L-C)' \\
& +\frac{(2\mathcal{G}'-\ln'(\mathcal{D}))(J-L)'}{2} + \frac{(J-L)''}{2} + \frac{(J-L)'}{r}] \\
& -4\sqrt{2}[\frac{2h'\sinh^2(h)}{r} - \frac{2\cosh(h)\sinh(h)}{r^2} + \frac{(J-L)'}{2r} + \frac{g'\cosh(h)\sinh(h)}{r}]\}
\end{aligned}$$

In appendix C, we give the completed equations for triplet ${}^3j_{j-1}$ and ${}^3j_{j+1}$ for general j.

Right now, we can apply the techniques which already developed for the Schrödinger-like system in nonrelativistic quantum mechanics to the above radial equations. We wish to compare this directly with Reid's potential and use it to fit the experimental phase shift data.

3.5 Relativistic Schrödinger Equation For Reid's Potential

The two body Dirac equation to be applied in this paper can be reduced to the Schrödinger-like equation(see Eq.(3.206))

$$(p^2 + \Phi(x_\perp))\Psi = b^2(w)\Psi$$

In the c.m. frame, this Schrödinger-like equation can be written in the familiar three dimensional Schrödinger form

$$(\mathbf{p}^2 + \Phi(r))\Psi = b^2(w)\Psi$$

We can rewrite this expression as

$$(\frac{\mathbf{p}^2}{2m_w} + \frac{\Phi(r)}{2m_w})\Psi = \frac{b^2(w)}{2m_w}\Psi$$

Comparing with the standard Schrödinger equation

$$(\frac{\mathbf{p}^2}{2m} + V(r))\Psi = E\Psi$$

we get

$$\frac{\mathbf{p}^2}{2m_w} \longrightarrow \frac{\mathbf{p}^2}{2m}, \qquad \frac{\Phi(r)}{2m_w} \longrightarrow V(r), \qquad \frac{b^2(w)}{2m_w} \longrightarrow E,$$

Define $2m_wV(r)$ as the relativistic Reid's potential. By comparing $\Phi(r)$ and $2m_wV(r)$

we can determine whether our $\Phi(r)$ is similar to Reid's potentials or not.

Chapter 4

Variable Phase Method

In this chapter, I will discuss and review the phase shift methods which we used in our numerical calculations, which include phase shift equations for uncoupled and coupled states and the phase shift equations with Coulomb potentials.

We discuss an approach[38] to the problem of evaluating the scattering phase shift produced by a spherically symmetrical potential. Actually, the phase shift equations which we used in this dissertation are slightly different from phase shift equations I talk about here because of the small r behavior of phase shift equations and our potentials. We modified the phase shift equations in some way to get our phase shift equations. In the chapter 5, I will talk about how to modify the phase shift equations to achieve our goal.

This method has several advantages over the traditional approach. In the traditional approach, one integrates the radial Schrödinger equation from the origin to the asymptotic region where the potential is negligible, and then compares the phase of the radial wave function with that of a free wave thus obtain the phase shift.

In this approach we need only integrate a first order non-linear differential equation from the origin to the asymptotic region, thereby obtaining directly the value of the scattering phase shift. This approach is termed the " variable phase method " or simply the "phase method" because the dependent variable on which we focus may always be interpreted as a scattering phase shift.

This method is a very convenient one for deriving general properties of the scattering phase shifts, and for performing numerical computations. A further advantage of this approach is its formal simplicity and the straightforward physical interpretation that may be given of its basic quantities and equations. In short, it is appears that the variable phase method is in all respects superior to the traditional method for studying scattering phase shifts. Most important for us, since we can reduce our two body Dirac equations to Schrödinger-like form, we can use this variable phase method to compute the phase shift for our relativistic two body equations.

4.1 Phase Shift Equation For Uncoupled Schrödinger Equation

For the scattering on a spherically symmetrical potential. The scattering situation is described by a stationary wave function , which is that solution of the time-independent Schrödinger equation[38],

$$[\nabla^2 + k^2 - V(r)]\psi(r) = 0, \tag{4.1}$$

characterized by the asymptotic boundary condition

$$\psi(r) \stackrel{r\to\infty}{\longrightarrow} e^{ikz} + f(\theta)\frac{e^{ikr}}{r}. \tag{4.2}$$

We have assumed here that the incident beam travels parallel to the z axis; the scattering angle θ is the angle between the direction of observation and the incident direction. The function $f(\theta)$ is the "scattering amplitude, " and it yields the differential cross section through

$$\frac{d\sigma(\theta)}{d\Omega} =\mid f(\theta) \mid^2 \tag{4.3}$$

Thus the scattering problem consists in the evaluation of $f(\theta)$. The symmetry of the problem is exploited by setting

$$\psi(r) = \frac{1}{r}\sum_{l=0}^{\infty} u_l(r) P_l(\cos\theta), \tag{4.4}$$

where the $P_l's$ are Legendre polynomials. Because $\psi(0)$ must be finite, this equation implies the boundary condition for $u_l(r)$ is

$$u_l(0) = 0 \tag{4.5}$$

Insert the Eq.(4.4) into Eq.(4.1) and we obtain the radial uncoupled Schrödinger equation

$$u_l''(r) + [k^2 - \frac{l(l+1)}{r^2} - V(r)]u_l(r) = 0 \tag{4.6}$$

The radial wave function thus is real, and it defines the " scattering phase shift " δ_l through the comparison of its asymptotic behavior with that of the sine function:

$$u_l(r) \overset{r\to\infty}{\longrightarrow} const \cdot \sin(kr - \frac{l\pi}{2} + \delta_l) \tag{4.7}$$

combining above equation with Eq.(4.2) , Eq.(4.4) and asymptotic equation,

$$e^{ikz} \overset{r\to\infty}{\longrightarrow} \frac{1}{kr}\sum_{l=0}^{\infty}(2l+1)i^l \sin(kr - \frac{l\pi}{2})P_l(\cos\theta),$$

we obtain for the scattering amplitude expression

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta). \tag{4.8}$$

We thus see that the scattering amplitude is expressed as a sum over the "partial wave amplitudes"

$$A_l = e^{i\delta_l}\sin\delta_l$$

We may write the integral equation satisfied by the radial wave function

$$u_l(r) = \hat{j}_l(kr) - \frac{1}{k}\int_0^r ds[\hat{j}_l(kr)\hat{n}_l(ks) - \hat{j}_l(ks)\hat{n}_l(kr)]V(s)u_l(s). \tag{4.9}$$

the Riccati-Bessel functions $\hat{j}_l(kr)$ and $\hat{n}_l(kr)$ are define in Appendix A, This integral equation is equivalent to the radial uncoupled Schrödinger equation (4.6).

Now, we review the derivation of the " phase equation ". First, we introduce two auxiliary functions, defined in terms of the radial wave function. They are

$$s_l(r) = -\frac{1}{k}\int_0^r dr' V(r')\hat{j}_l(kr')u_l(r')$$

$$c_l(r) = 1 - \frac{1}{k}\int_0^r dr' V(r')\hat{n}_l(kr')u_l(r')$$

from these definitions, and the behaviors of the functions $V(r)$, $\hat{j}_l(kr)$, $\hat{n}_l(kr)$ and $u_l(r)$, in the neighborhood of the origin, we infer

$$s_l(r) \xrightarrow{r\to 0} 0$$

$$c_l(r) \xrightarrow{r\to 0} 1$$

we can compare the definitions of two functions $s_l(r)$ and $c_l(r)$ with the integral equation satisfied by the radial wave function $u_l(r)$, Eq.(4.9); we obtain

$$u_l(r) = c_l(r)\hat{j}_l(kr) - s_l(r)\hat{n}_l(kr) \tag{4.10}$$

The asymptotic behavior of $u_l(r)$ is

$$u_l(r) \stackrel{r\to\infty}{\longrightarrow} c_l(\infty)\sin(kr - \frac{l\pi}{2}) + s_l(\infty)\cos(kr - \frac{l\pi}{2})$$

to obtain this expression, we have used the asymptotic expression of the Riccati-Bessel functions. A comparison of the above expression with Eq.(4.7), yields the equation

$$\tan\delta_l = \frac{s_l(\infty)}{c_l(\infty)}.$$

This relation suggests the introduction of a new function $t_l(r)$, defined by

$$t_l(r) = \frac{s_l(r)}{c_l(r)}.$$

This function vanishes at the origin exactly in the same manner as the function $s_l(r)$:

$$t_l(r) \stackrel{r\to 0}{\longrightarrow} 0 \tag{4.11}$$

the asymptotic behavior of function $t_l(r)$ yields directly the value of the tangent of the scattering phase shift:

$$\lim_{r\to\infty} t_l(r) \equiv t_l(\infty) = \tan\delta_l \tag{4.12}$$

At this point, we have learned that we may introduce a function $t_l(r)$ whose value at the origin is known, and whose asymptotic value yields directly the tangent of the scattering phase shift. Now the question is, is it possible to go from the origin to the asymptotic region ? One shows that the function $t_l(r)$ satisfies a first-order differential equation, so that once its value at the origin is known, we can obtain it everywhere; and we can evaluate its asymptotic value by integrating this differential equation from the origin to infinity. To derive this first-order differential equation of $t_l(r)$, we differentiate the equations that define the auxiliary functions $s_l(r)$ and $c_l(r)$, we also use Eq.(4.10) to substitute for $u_l(r)$ in the right hand side of equation. We thus obtain the following system of the two coupled first-order linear equations

$$s_l'(r) = -\frac{1}{k}V(r)\hat{j}_l(kr)[c_l(r)\hat{j}_l(kr) - s_l(r)\hat{n}_l(kr)]$$

$$c_l'(r) = -\frac{1}{k}V(r)\hat{n}_l(kr)[c_l(r)\hat{j}_l(kr) - s_l(r)\hat{n}_l(kr)]$$

Now we multiply the first equation by $c_l(r)$ and second by $s_l(r)$, subtract the second equation from the first, and divided by $c_l^2(r)$.We obtain

$$t_l'(r) = -\frac{1}{k}V(r)[\hat{j}_l(kr) - t_l(r)\hat{n}_l(kr)]^2 \tag{4.13}$$

This is our anticipated first-order nonlinear differential equation. Now we introduce another function $\delta_l(r)$ by

$$t_l(r) = \tan\delta_l(r) \tag{4.14}$$

Then from the Eq.(4.11) we also obtain for $\delta_l(r)$

$$\delta_l(r) \xrightarrow{r\to 0} 0$$

and from Eq.(4.12) we find that

$$\lim_{r\to\infty} \delta_l(r) \equiv \delta_l(\infty) = \delta_l$$

Inserting Eq.(4.14) into Eq(4.13), we find the differential equation for $\delta_l(r)$[38]

$$\delta_l'(r) = -k^{-1}V(r)\left[\cos\delta_l(r)\hat{j}_l(kr) - \sin\delta_l(r)\hat{n}_l(kr)\right]^2 \tag{4.15}$$

It is first order nonlinear differential equation and it yields asymptotically the value of the scattering phase shift.

$$\delta_l = \lim_{r\to\infty} \delta_l(r)$$

function $\delta_l(r)$ is named the " phase function ", Eq.(4.15) is called " phase equation ". It is our main tool for studying the properties of scattering phase shifts. The Eq.(4.15) become particular simple in the case of S waves, for S waves, it become

$$\delta_0'(r) = -k^{-1}V(r)\sin^2[kr + \delta_0(r)] \tag{4.16}$$

Eq.(4.15) can also be written in other forms

$$\delta_l'(r) = -k^{-1}V(r)\hat{D}_l^2(kr)\sin^2[\hat{\delta}_l(kr) + \delta_l(r)] \tag{4.17}$$

Where $\hat{D}_l^2(kr)$ and $\hat{\delta}_l(kr)$ are defined in Appendix A.

Now, since our Schrödinger-like equations in c.m. system has the form

$$[\nabla^2 - b^2 - \Phi]\psi = 0$$

we can directly follow the above steps to obtain the phase shift by swapping $k \to b$, and $V \to \Phi$. There is no change in phase shift equation, even though, our potential Φ depend on the c.m. system energy w.

4.2 Phase Shift Equation For Coupled Schrödinger Equation

For coupled Schrödinger equation, the phase shift equation has a more complicated form. We discuss an approach[39] in this section how to handle the coupled Schrödinger equation. Generally, the coupled(multi-channel) radial Schrödinger equation is in the form

$$\frac{d^2}{dr^2}\mathbf{\Psi}(r) + \mathbf{M}[2E - \frac{1}{r^2}\mathbf{L}\mathbf{M}^{-1} - 2\mathbf{V}(r)]\mathbf{\Psi}(r) = 0, \tag{4.18}$$

In this equation $\mathbf{M}$ is the diagonal mass matrix, L indicates the diagonal angular momentum matrix with elements $l_i(l_i+1)\delta_{ij}$ and $\mathbf{V}(r)$ is the potential matrix which we assume real and symmetric. The matrix elements satisfy the usual conditions:

$$\lim_{r\to\infty} rV_{ij}(r) = 0$$

$$\int_0^\infty r\,|V_{ij}(r)|\,dr < \infty$$

$$V_{ij}(r) \stackrel{r\to\infty}{\longrightarrow} V_{0ij}r^{\eta_{ij}}, \qquad \eta_{ij} \geq -1$$

Let $\Psi_\alpha(r)$ $(\alpha = 1, 2....n)$ be the n linear independent regular solutions of the n-channel differential Eq.(4.18) with the boundary conditions

$$\Psi_\alpha(0) = 0, \qquad\qquad (\alpha = 1, 2....n)$$

introduce

$$k_i^2 = 2m_iE, \qquad\qquad \zeta_i^\pm = \sqrt{\frac{m_i}{k_i}}\exp[\pm i(k_ir - \frac{l_i\pi}{2})]$$

and the asymptotic expression

$$\Psi_\alpha(r) \stackrel{r\to\infty}{\longrightarrow} \begin{pmatrix} B_{1\alpha}\zeta_1^-(r) - A_{1\alpha}\zeta_1^+(r) \\ ... \\ ... \\ B_{n\alpha}\zeta_n^-(r) - A_{n\alpha}\zeta_n^+(r) \end{pmatrix} \tag{4.19}$$

Let $\mathbf{A}$ and $\mathbf{B}$ be the constant matrices as introduced as through Eq(4.19). Then the $\mathbf{S}$ matrix of the scattering process is defined by

$$\mathbf{S} = \mathbf{A}\mathbf{B}^{-1}$$

With the help of the Riccati-Bessel functions(see Appendix A), one introduces the following diagonal matrices:

$$\begin{array}{lll} \widehat{J}(\mathbf{r}), & \text{with elements} & \delta_{ij}\sqrt{\frac{m_i}{k_i}}\widehat{\jmath}_{l_i}(k_i r) \\ \widehat{N}(\mathbf{r}), & \text{with elements} & \delta_{ij}\sqrt{\frac{m_i}{k_i}}\widehat{n}_{l_i}(k_i r) \end{array}$$

$$\mathbf{H}^{\pm}(r) = -\widehat{N}(\mathbf{r}) \pm i\widehat{J}(r)$$

Then the regular solution of $\boldsymbol{\Psi}(r)$ is

$$\boldsymbol{\Psi}(r) = \widehat{J}(\mathbf{r}) - 2\int_0^\infty [\widehat{J}(\mathbf{r})\widehat{N}(r') - \widehat{J}(r')\widehat{N}(\mathbf{r})]V(r')\Psi(r')dr'$$

Let us consider the following matrix

$$\mathbf{F}\ (r) = \mathbf{M}^{-1}[\mathbf{H}^{-}(r)\frac{d}{dr}(\mathbf{\Psi}(r)) - \frac{d}{dr}(\mathbf{H}^{-}(r))\mathbf{\Psi}(r)] \tag{4.20}$$

then from the integral Schrödinger equation it is easy to obtain the expression

$$\mathbf{F}(r) = \mathbf{I} + 2\int_{0}^{r}\mathbf{H}^{-}(r')V(r')\Psi(r')dr'$$

from which it follows that $F(0)$ is the unit matrix and $F(\infty)$ exists and is finite. So we consider the matrix

$$\mathbf{S}(r) = F(r)F^{*-1}(r).$$

is the scattering matrix and is then given by

$$\mathbf{S} = \lim_{r\to\infty}\mathbf{S}(r).$$

Using the Schrödinger equation and the following relations

$$\mathbf{\Psi}(r) = \frac{i}{2}(\mathbf{H}^{-}(r)F^{*}(r) - \mathbf{H}^{+}(r)F(r)),$$

$$\frac{d}{dr}F(r) = i\mathbf{H}^{-}(r)V(r)(\mathbf{H}^{-}(r)F^{*}(r) - \mathbf{H}^{+}(r)F(r))$$

one obtains the many channel **S**-matrix equation

$$\frac{d}{dr}\mathbf{S}(r) = i[(\mathbf{S}(r)\mathbf{H}^{+}(r) - \mathbf{H}^{-}(r))V(r)(\mathbf{H}^{+}(r)\mathbf{S}(r) - \mathbf{H}^{-}(r))] \tag{4.21}$$

with the boundary condition:

$$\mathbf{S}(0) = \mathbf{I}. \tag{4.22}$$

It is convenient to introduce three real parameters to define the eigenvector and eigenvalues of the matrix $\mathbf{S}$. Let δ_1, δ_2, ε be the three parameters defined by:

$$\mathbf{S}\mathbf{v}_i = \exp[2i\delta_i]\mathbf{v}_i \qquad\qquad i = 1, 2, \tag{4.23}$$

$$\mathbf{v}_1 = \begin{pmatrix} \cos\varepsilon \\ \\ \sin\varepsilon \end{pmatrix}, \qquad\qquad \mathbf{v}_2 = \begin{pmatrix} -\sin\varepsilon \\ \\ \cos\varepsilon \end{pmatrix} \tag{4.24}$$

We call δ_1 and δ_2 the phase shifts, and ε the mixing parameter. $\Psi_1(r)$ and $\Psi_2(r)$ are the two linearly independent solution of the radial Schrödinger equation with the asymptotic behaviors

$$\Psi_1(r) \overset{r\rightarrow\infty}{\longrightarrow} \begin{pmatrix} \sqrt{\frac{m_1}{k_1}}\cos\varepsilon\sin(k_1 r + \delta_1 - \frac{l_1\pi}{2}) \\ \sqrt{\frac{m_2}{k_2}}\sin\varepsilon\sin(k_2 r + \delta_2 - \frac{l_2\pi}{2}) \end{pmatrix},$$

$$\Psi_2(r) \overset{r\rightarrow\infty}{\longrightarrow} \begin{pmatrix} -\sqrt{\frac{m_1}{k_1}}\sin\varepsilon\sin(k_1 r + \delta_1 - \frac{l_1\pi}{2}) \\ \sqrt{\frac{m_2}{k_2}}\cos\varepsilon\sin(k_2 r + \delta_2 - \frac{l_2\pi}{2}) \end{pmatrix}.$$

We consider three function $\delta_1(r)$, $\delta_2(r)$, $\varepsilon(r)$ defined as in (4.23) and (4.24). In order to express these three real functions as solution of a system of differential equation it is more convenient to use the reactance matrix in place of the **S**-matrix:

$$\mathbf{T}(r) = i(\mathbf{I} - \mathbf{S}(r))(\mathbf{I} + \mathbf{S}(r))^{-1}. \tag{4.25}$$

Note from this equation and Eq(4.23) and Eq(4.24) we have

$$\mathbf{T}(r)\mathbf{v}_i(r) = t_i(r)\mathbf{v}_i(r), \qquad i = 1, 2, \tag{4.26}$$

with $t_i(r) = tg\delta_i(r)$.

From Eq(4.25) and **S**-matrix Eq(4.21) a real differential equation for the reactance-matrix is immediately obtained:

$$\frac{d}{dr}\mathbf{T}(r) = -2(\widehat{J}(\mathbf{r}) - \mathbf{T}(r)\widehat{N}(\mathbf{r}))V(r)(\widehat{J}(\mathbf{r}) - \widehat{N}(\mathbf{r})\mathbf{T}(r)) \tag{4.27}$$

with boundary conditions

$$\mathbf{T}(0) = 0. \tag{4.28}$$

Using the following relations

$$< \mathbf{v}_i(r), \mathbf{v}_j(r) >= \delta_{ij}, \tag{4.29}$$

$$\frac{d}{dr}\mathbf{v}_i(r) = (-1)^{i+1}\frac{d}{dr}\varepsilon(r)\sum_{j=1}^{2}(1-\delta_{ij})\mathbf{v}_j(r), \tag{4.30}$$

which may be immediately verified using the definition of $\mathbf{v}\ _i(r)$

$$\mathbf{v}_1(r) = \begin{pmatrix} \cos\varepsilon(r) \\ \\ \sin\varepsilon(r) \end{pmatrix}, \qquad \mathbf{v}_2(r) = \begin{pmatrix} -\sin\varepsilon(r) \\ \\ \cos\varepsilon(r) \end{pmatrix} \tag{4.31}$$

and taking into account the derivative of Eq(4.26) we can write

$$\left\langle \mathbf{v}_i(r), [\frac{d}{dr}\mathbf{T}(r)]\mathbf{v}_j(r) \right\rangle = \frac{d}{dr}t_i(r)\delta_{ik} + (-1)^{i+1}\frac{d}{dr}\varepsilon(r)\sum_{j=1}^{2}(t_i(r)-t_j(r))\delta_{jk}. \tag{4.32}$$

Introducing the matrices

$$\mathbf{I}(\mathbf{r}) = -\mathbf{2}\widehat{J}(\mathbf{r})V(r)\widehat{J}(\mathbf{r}),$$

$$\tilde{\mathbf{N}}(\mathbf{r}) = -\mathbf{2}\widehat{\mathbf{N}}(\mathbf{r})\mathbf{V}(\mathbf{r})\widehat{\mathbf{N}}(\mathbf{r}),$$

$$\mathbf{C}(\mathbf{r}) = \mathbf{2}\widehat{\mathbf{J}}(\mathbf{r})\mathbf{V}(\mathbf{r})\widehat{\mathbf{N}}(\mathbf{r}). \tag{4.33}$$

and using the notation $A_{ik} =< \mathbf{v}_i, \mathbf{A}\mathbf{v}_k >$ if $\mathbf{A}$ is one of the matrices of (4.33), we can obtain the following system of differential equations

$$\frac{d}{dr}t_i(r)\delta_{ik} + (-1)^{i+1}\frac{d}{dr}\varepsilon(r)\sum_{j=1}^{2}(t_i(r)-t_j(r))\delta_{jk} =$$

$$\mathbf{I}_{ki}(\mathbf{r})+t_i(r)\mathbf{C}_{ki}(\mathbf{r})+t_k(r)\mathbf{C}_{ik}(\mathbf{r})+t_k(r)t_i(r)\tilde{\mathbf{N}}_{ki}(\mathbf{r}). \tag{4.34}$$

For a simple case of equal masses and angular momenta in all the channel, the above differential equation becomes

$$\frac{d}{dr}t_1(r) = -\frac{2m}{k}(\cos^2\varepsilon(r)V_{11}(r)+\sin 2\varepsilon(r)V_{12}(r)+\sin^2\varepsilon(r)V_{22}(r))\cdot(\widehat{j}_l(kr)-t_1(r)\widehat{n}_l(kr))^2 \tag{4.35}$$

$$\frac{d}{dr}t_2(r) = -\frac{2m}{k}(\sin^2\varepsilon(r)V_{11}(r)-\sin 2\varepsilon(r)V_{12}(r)+\cos^2\varepsilon(r)V_{22}(r))\cdot(\widehat{j}_l(kr)-t_2(r)\widehat{n}_l(kr))^2 \tag{4.36}$$

$$\begin{aligned}[t_1(r)-t_2(r)]\frac{d}{dr}\varepsilon(r) &= \frac{2m}{k}[\sin 2\varepsilon(r)\frac{(V_{11}(r)-V_{22}(r))}{2} - \cos 2\varepsilon(r)V_{12}(r)]\cdot \\ &\quad (\widehat{j}_l(kr)-t_1(r)\widehat{n}_l(kr))(\widehat{j}_l(kr)-t_2(r)\widehat{n}_l(kr))\end{aligned} \tag{4.37}$$

We see from these last equations that at a distance r from the origin, the function $\varepsilon(r)$ gives through the linear combination

$$< \mathbf{v_i}(\mathbf{r}), \mathbf{V}(\mathbf{r})\mathbf{v_i}(\mathbf{r}) >= \begin{cases} \cos^2\varepsilon(r)V_{11}(r)+\sin 2\varepsilon(r)V_{12}(r)+\sin^2\varepsilon(r)V_{22}(r), & if\ i=1, \\ \sin^2\varepsilon(r)V_{11}(r)-\sin 2\varepsilon(r)V_{12}(r)+\cos^2\varepsilon(r)V_{22}(r), & if\ i=2, \end{cases}$$

of the element of the potential matrix, an equivalent potential related to the process in the i-th channel as it appears in the one-channel equation for the tangent of the phase shift. A system of differential equations for the phase shift can be derive easily from above differential equation

$$\frac{d}{dr}\delta_i(r) = -\frac{2m}{k} < \mathbf{v}_i(r), \mathbf{V}(r)\mathbf{v}_i(r) > \widehat{D}_l^2(kr)\sin^2(\widehat{\delta}_l(kr)+\delta_i(r)), \qquad i=1,2,$$

$$\sin(\delta_1(r)-\delta_2(r))\frac{d}{dr}\varepsilon(r) = -\frac{2m}{k} < \mathbf{v}_1(r), \mathbf{V}(r)\mathbf{v}_2(r) > \cdot\widehat{D}_l^2(kr)\sin(\widehat{\delta}_l(kr)+\delta_1(r))\times$$
$$\sin(\widehat{\delta}_l(kr)+\delta_2(r)) \tag{4.38}$$

where the function $\widehat{D}_l(kr)$ and $\widehat{\delta}_l(kr)$ are the amplitude and phase shift of the Riccati-Bessel functions as defined in Appendix A. In the general case, a more complicated system of differential equations are found. They are[39]

$$\begin{aligned}
&\frac{d}{dr}\delta_1(r) \\
= \quad & -\frac{1}{\sqrt{E}}(1+\mu^2(r))^{-1}\{\sqrt{2m_1}V_{11}(r)\widehat{D}_{l_1}^2(k_1r)\sin^2(\widehat{\delta}_{l_1}(k_1r)+\delta_1(r)) \\
& +2\sqrt{2\sqrt{m_1m_2}}\widehat{D}_{l_1}(k_1r)\widehat{D}_{l_2}(k_2r)V_{12}(r)\mu(r)(\sin\delta_1(r)\sin(\widehat{\delta}_{l_1}(k_1r)+\widehat{\delta}_{l_2}(k_2r)+\delta_1(r)) \\
& +\sin\widehat{\delta}_{l_1}(k_1r)\sin\widehat{\delta}_{l_2}(k_2r))+\sqrt{2m_2}V_{22}(r)\mu^2(r)\widehat{D}_{l_2}^2(k_2r)\sin^2(\widehat{\delta}_{l_2}(k_2r)+\delta_1(r))\}
\end{aligned} \tag{4.39}$$

$$\frac{d}{dr}\delta_2(r)$$

$$
\begin{aligned}
= \quad & -\frac{1}{\sqrt{E}}(1+\mu^2(r))^{-1}\{\sqrt{2m_1}V_{11}(r)\mu^2(r)\widehat{D}_{l_1}^2(k_1r)\sin^2(\widehat{\delta}_{l_1}(k_1r)+\delta_2(r)) \\
& -2\sqrt{2\sqrt{m_1m_2}}\widehat{D}_{l_1}(k_1r)\widehat{D}_{l_2}(k_2r)V_{12}(r)\mu(r)(\sin\delta_2(r)\sin(\widehat{\delta}_{l_1}(k_1r)+\widehat{\delta}_{l_2}(k_2r)+\delta_2(r)) \\
& +\sin\widehat{\delta}_{l_1}(k_1r)\sin\widehat{\delta}_{l_2}(k_2r))+\sqrt{2m_2}V_{22}(r)\widehat{D}_{l_2}^2(k_2r)\sin^2(\widehat{\delta}_{l_2}(k_2r)+\delta_2(r))\}
\end{aligned}
\tag{4.40}
$$

$$
\begin{aligned}
& \sin(\delta_1(r)-\delta_2(r))\frac{d}{dr}\mu(r) \\
= \quad & \frac{1}{\sqrt{E}}\{\sqrt{2m_1}V_{11}(r)\mu(r)\widehat{D}_{l_1}^2(k_1r)\sin(\widehat{\delta}_{l_1}(k_1r)+\delta_1(r))\sin(\widehat{\delta}_{l_1}(k_1r)+\delta_2(r)) \\
& -\sqrt{2m_2}V_{22}(r)\mu(r)\widehat{D}_{l_2}^2(k_2r)\sin(\widehat{\delta}_{l_2}(k_2r)+\delta_1(r))\sin(\widehat{\delta}_{l_2}(k_2r)+\delta_2(r)) \\
& +2\sqrt{2\sqrt{m_1m_2}}\widehat{D}_{l_1}(k_1r)\widehat{D}_{l_2}(k_2r)V_{12}(r)[(\mu^2(r)-1)\sin\widehat{\delta}_{l_1}(k_1r)\sin\widehat{\delta}_{l_2}(k_2r)\cos\delta_1\cos\delta_2 \\
& +\sin\delta_1(r)\cos\delta_2(r)(\mu^2(r)\cos\widehat{\delta}_{l_2}(k_2r)\sin\widehat{\delta}_{l_1}(k_1r)-\cos\widehat{\delta}_{l_1}(k_1r)\sin\widehat{\delta}_{l_2}(k_2r)) \\
& +\sin\delta_2(r)\cos\delta_1(r)(\mu^2(r)\cos\widehat{\delta}_{l_1}(k_1r)\sin\widehat{\delta}_{l_2}(k_2r)-\cos\widehat{\delta}_{l_2}(k_2r)\sin\widehat{\delta}_{l_1}(k_1r))
\end{aligned}
$$

$$+\sin\delta_1(r)\sin\delta_2(r)(\mu^2(r)-1)\cos\widehat{\delta}_{l_1}(k_1r)\cos\widehat{\delta}_{l_2}(k_2r)]\} \tag{4.41}$$

where we have

$$\mu(r) = tg\varepsilon(r)$$

4.3 Phase Shift Equation With The Coulomb Potential

We review here the necessary modification of our phase equations when we consider the *pp* scattering[25]. When study the *pp* scattering, we must consider the Coulomb potential. The general form of the uncoupled Schrödinger-like equation with Coulomb potential is[25]

$$\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}-\frac{2\epsilon_w\alpha}{r}+\Delta\Phi\right]u(r)=b^2u(r), \tag{4.42}$$

where $\Delta\Phi$ consists of the short range parts of the effective potential. α is the fine structure constant. Due to the long range behavior of the potential in above equation, the asymptotic behavior of the wave function is

$$u(r) \stackrel{r\to\infty}{\longrightarrow} const\cdot\sin(br-\eta\ln 2br+\Delta), \tag{4.43}$$

in which

$$\Delta = \delta_l + \sigma_l - \frac{l\pi}{2},$$

where $\sigma_l = \arg\Gamma(l+1+i\eta)$ is the Coulomb phase shift, here $\eta = -\frac{\epsilon_w \alpha}{b}$.

We also use the variable phase method to calculate the phase shift with Coulomb potential. Consider two differential equations

$$u'' + (b^2 - W - \overline{W})u = 0,$$

and

$$\overline{u}'' + (b^2 - \overline{W})\overline{u} = 0, \qquad i = 1, 2$$

in which $u(0) = \overline{u}_1(0) = 0$. Let

$$\overline{W}(r) = -\frac{2\epsilon_w \alpha}{r},$$

$$W(r) = \frac{l(l+1)}{r^2} + \Delta\Phi,$$

so that

$$\overline{u}_1(r) \overset{r\to\infty}{\longrightarrow} const \cdot \sin(br - \eta \ln 2br + \overline{\Delta}),$$

$$\overline{u}_2(r) \overset{r\to\infty}{\longrightarrow} const \cdot \cos(br - \eta \ln 2br + \overline{\Delta}),$$

where $\overline{\Delta} = \sigma_0$.

Just as in the variable phase method, we obtain a nonlinear first order differential equation for the phase shift function $\delta_l(r)$ such that $\delta_l(\infty) = \delta_l$, and $\delta_l(0) = 0$. This is done by rewriting $u(r)$ as

$$u(r) = \alpha(r)[\cos\gamma(r)\overline{u}_1(r) + \sin\gamma(r)\overline{u}_2(r)]$$

so that

$$\Delta = \overline{\Delta} + \gamma(\infty).$$

Since we have rewritten $u(r)$ in two arbitrary functions, we are free to impose a condition on $u(r)$

$$u'(r) = \alpha'(r)[\cos\gamma(r)\overline{u}_1'(r) + \sin\gamma(r)\overline{u}_2'(r)].$$

Combining $u(r)$ and $u'(r)$ leads to

$$\gamma(r) = -\tan^{-1}[\frac{u(r)\overline{u}_1'(r) - u'(r)\overline{u}_1(r)}{u(r)\overline{u}_2'(r) - u'(r)\overline{u}_2(r)}]$$

where $\gamma(0) = 0$, $\overline{u}_1(r) = F_0(\eta, br)$ and $\overline{u}_2(r) = G_0(\eta, br)$. With the Wronskian $F_0G_0' - F_0'G_0 = b$, we obtain by differentiating the differential equation

$$\gamma'(r) = -W(r)[\cos\gamma(r)F_0(\eta, br) + \sin\gamma(r)G_0(\eta, br)]^2/b$$

Note that for

$$W(r) \xrightarrow{r\to 0} \frac{\lambda(\lambda+1)}{r^2}, \qquad \frac{\lambda(\lambda+1)}{r^2} = \frac{l(l+1)}{r^2} - \frac{\alpha^2}{r^2},$$

$$F_0(\eta, br) \xrightarrow{r\to 0} C_0 br,$$

$$G_0(\eta, br) \xrightarrow{r\to 0} \frac{1}{C_0},$$

we obtain the relation

$$\gamma'(0) = -\frac{C_0^2 b\lambda}{\lambda(\lambda+1)}.$$

Letting

$$\gamma(r) = \beta(r) + \epsilon(r),$$

where $\beta(r)$ is defined as

$$\beta'(r) = -\frac{l(l+1)}{r^2}[\cos\gamma(r)F_0(\eta, br) + \sin\gamma(r)G_0(\eta, br)]^2/b$$

$\beta(r)$ has the exact solution

$$\gamma(r) = -\tan^{-1}[\frac{F_l(\eta, br)F_0'(\eta, br) - F_l'(\eta, br)F_0(\eta, br)}{F_l(\eta, br)G_0'(\eta, br) - F_l'(\eta, br)G_0(\eta, br)}]$$

with $\beta(0) = 0$ and $\beta'(0) = -\frac{C_0^2 bl}{l(l+1)}$ and $\beta(\infty) = \sigma_l - \frac{l\pi}{2} - \sigma_0$, lead to

$$\delta_l = \epsilon(\infty).$$

Thus , if we solve

$$\begin{aligned}\epsilon'(r) \quad &= \quad [-\frac{l(l+1)}{r^2} + \Delta\Phi][\cos(\beta(r) + \epsilon(r))F_0(\eta, br) + \sin(\beta(r) + \epsilon(r))G_0(\eta, br)]^2/b \\ &\quad + \frac{l(l+1)}{r^2}[\cos\beta(r)F_0(\eta, br) + \sin\beta(r)G_0(\eta, br)]^2/b\end{aligned}$$

with the condition $\epsilon(0) = 0$, we obtain the additional phase shift(above the Coulomb phase shift) by integration to $\epsilon(\infty)$.

There is no Coulomb scattering for the triplet $^3S_1$ and $^3D_1$ states as a consideration of Pauli principal would show.

Chapter 5

Model

In this chapter, I will talk about the model we used in our calculation which include how to model our $\mathcal{A}$, S and C invariant potential functions, the mesons we used in our calculation and the way they enter into the two body Dirac equations, and how we modify the phase shift equation for singlet and triplet states.

5.1 Two Body Dirac Equations For Scalar And Vector Interactions

The Dirac equations[17,18,19,21,23] of constraint dynamics for two relativistic spin-one-half particle interacting through scalar and vector potentials are

$$\mathcal{S}_1\psi \equiv \gamma_{51}(\gamma_1 \cdot (p_1 - A_1) + m_1 + S_1)\psi = 0 \tag{5.1}$$

$$\mathcal{S}_2\psi \equiv \gamma_{52}(\gamma_2 \cdot (p_2 - A_2) + m_2 + S_2)\psi = 0 \tag{5.2}$$

the subscripts $i = 1, 2$ stands for the i th particles, so m_1 and m_2 are masses of two interacting fermionic particles. A_i^μ and S_i introduce the interactions that the i th particle experience due to the presence of the other particle, we call this form of the two body Dirac equations in the " external potential forms " . In QCD, the scalar potential S_i is semi-phenomenological, the vector potential A_i^μ is semi-phenomenological in the long range and in the short range are closely related to the perturbative quantum field theory. $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfy the compatibility condition[31]

$$[\mathcal{S}_1, \mathcal{S}_2] = 0. \tag{5.3}$$

The vector potential A_i^μ are given in terms of three invariant functions[20,21,23] G, E_1, E_2 by

$$A_1^\mu = ((\epsilon_1 - E_1) - i\frac{G}{2}\gamma_2 \cdot (\frac{\partial E_1}{E_2} + \partial \ln G)\gamma_2 \cdot \widehat{P})\widehat{P}^\mu + (1 - G)p^\mu - \frac{i}{2}\partial G \cdot \gamma_2 \gamma_2^\mu \tag{5.4}$$

$$A_2^\mu = ((\epsilon_2 - E_2) - i\frac{G}{2}\gamma_1 \cdot (\frac{\partial E_2}{E1} + \partial \ln G)\gamma_1 \cdot \widehat{P})\widehat{P}^\mu + (1 - G)p^\mu - \frac{i}{2}\partial G \cdot \gamma_1 \gamma_1^\mu \tag{5.5}$$

while the scalar potentials S_i are given in terms of three invariant functions[20,21,23] G, M_1, M_2 by

$$S_1 = M_1 - m_1 - \frac{i}{2}G\gamma_2 \cdot \frac{\partial M_1}{M_2} \tag{5.6}$$

$$S_2 = M_2 - m_2 - \frac{i}{2} G\gamma_1 \cdot \frac{\partial M_2}{M_1} \tag{5.7}$$

In Eq(5.4) to Eq(5.5), the variable $P = p_1 + p_2$ is the total four momentum, $-P^2 = w^2$ is the c.m. energy squared, so $\widehat{P}^2 = -1$, where $\widehat{P} \equiv \frac{P}{w}$. The variable ϵ_i are the conserved c.m. energies of the constituent particles given by

$$\epsilon_1 = \frac{w^2 + m_1^2 - m_2^2}{2w}, \qquad \epsilon_2 = \frac{w^2 + m_2^2 - m_1^2}{2w} \tag{5.8}$$

so $w = \epsilon_1 + \epsilon_2$. The relative momentum defined by $p_1 = \epsilon_1 \hat{P} + p$, $p_2 = \epsilon_2 \hat{P} - p$ become

$$p = (\epsilon_2 p_1 - \epsilon_1 p_2)/w.$$

In order that Eq.(5.1) and Eq.(5.2) satisfy Eq.(5.3), it is necessary that the invariant functions G, E_1, E_2, M_1 and M_2 depend on the relative separation, $x = x_1 - x_2$, only through the space-like coordinate four vector

$$x_\perp^\mu = x_\mu + \widehat{P}^\mu (\widehat{P} \cdot x)$$

$x_\perp^\mu$ is perpendicular to the total four-momentum P. For QCD application, G, E_1, E_2 are functions[17,19,23]of $\mathcal{A}$. The forms for functions E_1, E_2, G are

$$E_1 = G(\epsilon_1 - \mathcal{A}) \tag{5.9}$$

$$E_2 = G(\epsilon_2 - \mathcal{A}) \tag{5.10}$$

and

$$G^2 = \frac{1}{(1 - \frac{2\mathcal{A}}{w})} \tag{5.11}$$

function $\mathcal{A}(r)$ is responsible for the covariant electromagnetic-like parts of A_i^μ. Even though the dependence of E_1, E_2, G on $\mathcal{A}$ is not unique, they are constrained by the requirement that they yield the correct nonrelativistic and semi-relativistic limits. For QCD application, M_1 and M_2 are the function of two invariant functions[18,23], $\mathcal{A}(r)$ and $S(r)$

$$M_1^2(\mathcal{A}, S) = m_1^2 + G^2(2m_w S + S^2) \tag{5.12}$$

$$M_2^2(\mathcal{A}, S) = m_2^2 + G^2(2m_w S + S^2). \tag{5.13}$$

The invariant function $S(r)$ is responsible for the scalar potential since $S_i = 0$, if $S(r) = 0$, while $\mathcal{A}(r)$ contribute to the S_i(if $S(r) \neq 0$) as well as to the vector potential A_i^μ. Finally, the five invariant functions G, E_1, E_2, M_1 and M_2 depend on two independent invariant functions S and $\mathcal{A}$. The kinematical variables m_w and ϵ_w are relativistic reduced mass for fictitious particle of relative motion(see Eq(3.50) to Eq(3.57)),

$$m_w = \frac{m_1 m_2}{w} \tag{5.14}$$

and energy of fictitious particle of relative motion,

$$\epsilon_w = \frac{w^2 - m_1^2 - m_2^2}{2w} \tag{5.15}$$

the corresponding value of the on-mass-shell relative momentum squared takes the form

$$p^2 = \epsilon_w^2 - m_w^2 = \frac{w^4 - 2w^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2}{4w^2} = b^2(w) = \epsilon_1^2 - m_1^2 = \epsilon_2^2 - m_2^2$$

The two body Dirac equations are two body counterparts of the one body Dirac equation with eliminated field and possess an analogous inherited dynamical " gauge invariance". In fact, Eq.(5.1) and Eq.(5.2) are invariant under any gauge transformation of the form $A_i^\mu \to A_i^\mu + \partial_i^\mu \chi(x_\perp)$ with χ the phase change of the single wave function.

To express G, E_1, E_2, M_1 and M_2 in terms of S and $\mathcal{A}$ are important for semi-phenomenological and other applications that emphasize the relationship of the interactions to external potentials of the two associated one-body problems. However, five invariants G, E_1, E_2, M_1 and M_2 can be expressed in the hyperbolic representation[24] in terms of L, J and $\mathcal{G}$. This representation is(for detail see chapter 3)

$$M_1 = m_1 \ \mathrm{ch}(L) \ + m_2 \ \mathrm{sh}(L)\,, \tag{5.16}$$

$$M_2 = m_2 \ \mathrm{ch}(L) \ + m_1 \ \ \mathrm{sh}(L)\,, \tag{5.17}$$

$$E_1 = \epsilon_1 \ \mathrm{ch}(J) \ + \epsilon_2 \mathrm{sh}(J)\,, \tag{5.18}$$

$$E_2 = \epsilon_2 \ \mathrm{ch}(J) \ + \epsilon_1 \ \mathrm{sh}(J)\,, \tag{5.19}$$

$$G = e^{\mathcal{G}}, \tag{5.20}$$

L, J and $\mathcal{G}$ generate scalar, time-like vector and space-like vector interactions respectively. If we use Eq.(5.16) to Eq.(5.20) and the " theta" matrices

$$\theta_i^{\mu} = i\sqrt{\frac{1}{2}}\gamma_{5i}\gamma_i^{\mu}, \qquad \mu = 0,1,2,3, \qquad i = 1,2 \tag{5.21}$$

$$\theta_{5i} = i\sqrt{\frac{1}{2}}\gamma_{5i} \tag{5.22}$$

we can put Eq.(5.1) and Eq.(5.2) in the form[25]

$$\mathbf{S}_1\psi = (\mathcal{S}_{10}ch(\Delta) + \mathcal{S}_{20}sh(\Delta))\psi = 0 \tag{5.23}$$

$$\mathbf{S}_2\psi = (\mathcal{S}_{20}ch(\Delta) + \mathcal{S}_{10}sh(\Delta))\psi = 0 \tag{5.24}$$

where

$$\Delta = \Delta_L + \Delta_J + \Delta_{\mathcal{G}}$$

and for scalar interactions

$$\Delta_L = -\frac{\mathcal{O}_1 L(x_\perp)}{2} = -\frac{I_1 I_2 L(x_\perp)}{2}\mathcal{O}_1, \qquad \mathcal{O}_1 = 2\theta_{51}\theta_{52} \tag{5.25}$$

I_1 and I_2 are identity operators. For time-like vector interactions

$$\Delta_J = \frac{\mathcal{O}_2 J(x_\perp)}{2} = \frac{\gamma_1\cdot \hat{P}\, \gamma_2\cdot \hat{P}\, J(x_\perp)}{2}\mathcal{O}_1 \qquad \mathcal{O}_2 = 2\theta_1\cdot\hat{P}\theta_2\cdot\hat{P} \tag{5.26}$$

for space like vector interactions

$$\Delta_{\mathcal{G}} = \frac{\mathcal{O}_3 \mathcal{G}(x_\perp)}{2} = \frac{\gamma_{1\perp}\cdot\gamma_{2\perp}\mathcal{G}(x_\perp)}{2}\mathcal{O}_1 \qquad \mathcal{O}_3 = 2\gamma_{1\perp}\cdot\gamma_{2\perp} \tag{5.27}$$

We may use Eq.(5.23) and Eq.(5.24) to relate the matrix potentials Δ to a given field theoretical or semi-phenomenological Feynman amplitude. A matrix amplitude proportional to $\gamma_1^\mu\cdot\gamma_{2\mu}$ corresponding to an electromagnetic-like interaction would indicate[25] $J = -\mathcal{G}$. Matrix amplitude proportional to either $I_1 I_2$ or $\gamma_1\cdot \hat{P}\, \gamma_2\cdot \hat{P}$ would correspond to semi-phenomenological scalar or time-like vector interactions. The two body Dirac equations in hyperbolic form, Eq.(5.23) and Eq.(5.24), give a simple version[24] for the norm of the sixteen component Dirac spinor. The two body Dirac equations in " external potential " form , Eq.(5.1) and Eq.(5.2), are simpler to reduce to the Schrödinger-like form and are useful for numerical calculation(see however Sazdjian[27] for a related reduction).

5.2 Mesons In Our Fitting

We obtain our semi-phenomenological potentials of two nucleon interaction by incorporating the meson exchange model and the two body Dirac equations. We put all the possible mesons which have physical meaning in nucleon-nucleon scattering in our fitting. The pion mediates the long range part of our semi-phenomenological potentials. Because the pion is the lightest meson, its exchange is associated with the longest range nuclear force. The short range behavior of our semi-phenomenological potentials are modified by the form factors, which are treated purely phenomenologically, because we do not have a very good theory about quarks and gluons. So there is no reason to include the heavy mesons that mediate the range which is shorter than the range modified by the form factors. The intermediate range part of our semi-phenomenological potentials comes mainly from exchange of mesons which are heavier than the pion. We use a total of 9 mesons in our fitting, these include scalar mesons σ, a_0 and f_0, vector mesons ρ, ω and ϕ, and pseudoscalar mesons π, η and η'. In this dissertation, we are ignoring tensor and pseudovector interactions, limiting ourselves to vector, scalar and pseudoscalar interactions, all the mesons we used have masses less than about 1000 MeV. See the Table 2.1 in chapter 2 for the detail features of the mesons we used.

Our scalar interactions enter into two body Dirac equations in the form(see Eq(5.6), Eq(5.7), Eq(5.12), Eq(5.13))

$$S = -g_\sigma^2 \frac{e^{-m_\sigma r}}{r} - (\tau_1 \cdot \tau_2) g_{a_0}^2 \frac{e^{-m_{a_0} r}}{r} - g_{f_0}^2 \frac{e^{-m_{f_0} r}}{r} \tag{5.28}$$

where g_σ^2, $g_{a_0}^2$, $g_{f_0}^2$ are coupling constants for mesons σ, a_0 and f_0. m_σ, m_{a_0} and m_{f_0} are the masses for corresponding meson, $(\tau_1 \cdot \tau_2)$ is 1 or 3 for isospin triplet or singlet states.

Our pseudoscalar interactions enter into two body Dirac equations in the form(see Eq(3.175))

$$C = (\tau_1 \cdot \tau_2)\frac{g_\pi^2}{w}\frac{e^{-m_\pi r}}{r} + \frac{g_\eta^2}{w}\frac{e^{-m_\eta r}}{r} + \frac{g_{\eta'}^2}{w}\frac{e^{-m_{\eta'} r}}{r}, \tag{5.29}$$

where $w = \epsilon_1 + \epsilon_2$ is total energy of two nucleon system. g_π^2, g_η^2, $g_{\eta'}^2$ are coupling constants for mesons π, η and η' respectively, m_π, m_η and $m_{\eta'}$ are the masses for corresponding mesons. We chosen this form for C, because it yield the correct nonrelativistic limit at low energy.

Our vector interactions enter into two body Dirac equations in the form(see Eq(5.9) to Eq(5.11))

$$A = (\tau_1 \cdot \tau_2)g_\rho^2\frac{e^{-m_\rho r}}{r} + g_w^2\frac{e^{-m_w r}}{r} + g_\phi^2\frac{e^{-m_\phi r}}{r} \tag{5.30}$$

where g_ρ^2, g_ω^2, g_ϕ^2 are coupling constants for mesons ρ, ω and ϕ. m_ρ, m_ω and m_ϕ are the masses for corresponding mesons.

We use the form factors to modify the small r behaviors in S, C and A. The meson exchange model is used to describe the nature of the nucleon-nucleon scattering in the long range and the intermediate range, but is not sensitive to the short range behavior. We use the form factors to modify short range part of nucleon-nucleon interaction. The final results should be independent on the detail of how the form factors are chosen. We choose our form factors by modifying small r in S, C and A to

$$r \longrightarrow \sqrt{r^2 + r_0^2}. \tag{5.31}$$

In our first model, we just use two different r_0 to fit the experimental data, one r_0 for pion, one for all the other 8 mesons whose masses are heavier than pion's mass. We set our two r_0 as two free parameters in our fitting.

In the constraint equations, $\mathcal{A}$ and S are relativistic invariant functions of the invariant separation $r = \sqrt{x_\perp^2}$. It is possible that $\mathcal{A}$ and S as identified from the nonrelativistic limit can take on all value between positive and negative infinity. So it is necessary to modify G, E_1, E_2, M_1 and M_2 so that the interaction functions remain real when one of the masses becomes very large or when $\mathcal{A}$ become large and repulsive[44]. These modifications are not unique but must maintain correct limits.

We have tested several models, two of which can give us fair good fit to the experimental data.

Model 1 For $E_i = G(\epsilon_i - \mathcal{A})$ to be real, we need only require that G be real or $\mathcal{A} < w/2$. This restriction on $\mathcal{A}$ is enough to ensure that $M_i = G\sqrt{m_i^2(1 - 2\mathcal{A}/w) + 2m_w S + S^2}$ be real as well(as long as $S \geq 0$). In order that $\mathcal{A}$ be so restricted we choose to redefine it as

$$\mathcal{A} = A, \qquad A \leq 0 \tag{5.32}$$

$$\mathcal{A} = \frac{A}{\sqrt{4A^2 + w^2}}, \qquad A \geq 0. \tag{5.33}$$

This parametrization gives an $\mathcal{A}$ that is continuous through its first derivative.

We next consider the problems that may arise in the limit when one of the masses becomes very large[44]. We must modify M_1 and M_2 so that it has the correct static limit(say $m_2 \to \infty$). It does appear that $M_1 \to m_1 + S$ when $m_2 \to \infty$. However this is only true if $m_1 + S \geq 0$. In the other word, in the limit $m_2 \to \infty$, the two body Dirac equations would be reduced to

$$(\gamma \cdot p_1 + \mid m_1 + S \mid)\psi = 0.$$

this would deviate from the standard Dirac equation in the region of strong attractive scalar potential ($S < -m_1$). In order to correct this problem, we take advantage of the hyperbolic parametrization. We desire a form for M_i that has the expected behavior ($M_i \to m_i + S$ in the limit when S becomes large and negative and one of the masses is large). So we modify our L in the following way[44]

$$shL = \frac{S}{WD}(1.0 + \frac{(\epsilon_w - A)S}{m_w D\sqrt{w^2 + s^2}}), \qquad S < 0 \tag{5.34}$$

where

$$D = 1 - \frac{2A}{w}$$

and for

$$S > 0$$

$$M_1^2 = m_1^2 + G^2(2m_w S + S^2)$$

$$M_2^2 = m_2^2 + G^2(2m_w S + S^2) \tag{5.35}$$

with Eq(5.16), Eq(5.17) and

$$\partial L = \frac{\partial M_1}{M_2}$$

A crucial feature of the shL extrapolation is that for fixed S, the static limit($m_2 \gg m_1$) form is $shL \to S/w$ which leads to $M_1 \to m_1 + S$. The above modifications are not unique, we may choose other way to modify our $\mathcal{A}$ and L, but the above modifications give us best fit up to now.

Model 2 This model come from the work of H. Sazdjian[43]. Using the special techniques, he is able to sum an infinite number of Feynman diagrams(of the ladder and cross ladder variety). For the vector interactions, he obtained results that correspond to Eq.(3.47) to Eq.(3.49) and Eq.(5.9) to Eq.(5.11)(modify here in Eq.(5.32)) for $A \geq 0$). For scalar interactions ($L(S, A)$) he obtained two results. One again agrees with Eq.(3.45) and Eq.(5.12) to Eq.(5.13). As we have seen above this must be modified(see Eq.(5.34)) for $S \leq 0$. His second result is one that give us another model for ($L(S, A)$) that would replace Eq.(5.12) to Eq.(5.13) with Eq.(5.36). We use this as our second model modified as below

If

$$S + A > 0$$

we let

$$S \longrightarrow -\mathcal{A} + \frac{(S+A)w}{\sqrt{4(S+A)^2 + w^2}}$$

If

$$S + A < 0$$

we let

$$S \longrightarrow -\mathcal{A} + S + A$$

and we let

$$shL = \sinh(-\frac{1}{2}\ln(1 - \frac{2(S+\mathcal{A})}{w}) - \mathcal{G}). \quad (5.36)$$

5.3 Non-minimal Coupling Of Vector Mesons

We mention that the coupling of the vector mesons in Eq(5.30) corresponds in quantum field theory to the renormalizable minimal coupling $g_\rho V_\mu \overline{\psi}\gamma^\mu \ \psi$ that is analogous to $eA_\mu \overline{\psi}\gamma^\mu \ \psi$ in QED. In our model, we are not concerned about renormalization, since the quantum field theory is not fundamental, so that we can not rule out the non-minimal coupling of the ρ, ω, ϕ analogous to

$$i\frac{e}{2M}\bar{\psi}[\gamma^\mu, \gamma^\nu]\psi F_{\mu\nu}, \quad (5.37)$$

We can convert above expression to something simpler[33]. By integration by part and using the free Dirac equation for the spinor field, this non renormalizable interaction can be converted into

$$i\frac{e}{2M}\bar{\psi}[\gamma^\mu,\gamma^\nu]\psi F_{\mu\nu} \rightarrow -i\frac{4em_N}{M}\bar{\psi}\gamma^\mu\psi A_\mu - i\frac{2e}{M}(\bar{\psi}\partial^\mu\psi - (\partial^\mu\bar{\psi})\psi)A_\mu \tag{5.38}$$

The first term can be put into the standard coupling while the second term gives rise to an amplitude written below. Now changing from photon to vector mesons (ρ) and using on shell features

$$\frac{4f_\rho^2(\eta_{\mu\nu}+\frac{q_\mu q_\nu}{m_\rho^2})(p+p')^\mu(p+p')^\nu}{M^2(q^2+m_\rho^2-i\varepsilon)} = \frac{4f_\rho^2(p+p')^2}{M^2(q^2+m_\rho^2-i\varepsilon)} = \frac{-4f_\rho^2(4m_N^2+q^2)}{M^2(q^2+m_\rho^2-i\varepsilon)} \tag{5.39}$$

where $q = p - p'$. The mass M is a mass scale for the interaction, m_N is the fermion (nucleon) mass and m_ρ is the ρ meson mass.

How does this interaction modify our Dirac equations? Which of the 8 or so invariants are effected(see Eq(3.100) to Eq(3.111))? In terms of its matrix structure, the above would appear to contribute to what we called Δ_L (see Eq(3.100)). It is as if we include an additional scalar interaction with an exchanged mass of a ρ and subtract from it the Laplacian(the q^2 terms in Eq(5.39)). That is

$$S \rightarrow S - \nabla^2 S/4m_N^2$$

where

$$S = -\frac{16m_N^2}{M^2}\frac{f_\rho^2 \exp(-m_\rho r)}{r}$$

Ignoring the delta function would give

$$S \to S(1 - m_\rho^2/4m_N^2) \tag{5.40}$$

so that the modification is rather simple. It appears to have the opposite sign as the vector interaction. That is, it would produce an attractive interaction for *pp* scattering. In our application, this mean the Eq(5.28) and Eq(5.30) are replaced(including the r_0 in Eq(5.31)) by

$$S = -g_\sigma^2\frac{e^{-m_\sigma \overline{r}}}{\overline{r}} - g_{a_0}^2\frac{e^{-m_{a_0}\overline{r}}}{\overline{r}} - g_{f_0}^2\frac{e^{-m_{f_0}\overline{r}}}{\overline{r}} - S' \tag{5.41}$$

where

$$S' = (\tau_1 \cdot \tau_2)g_\rho'^2(1 - \frac{\nabla^2}{4m_N^2})\frac{e^{-m_\rho \overline{r}}}{\overline{r}} + g_w'^2(1 - \frac{\nabla^2}{4m_N^2})\frac{e^{-m_w \overline{r}}}{\overline{r}} + g_\phi'^2(1 - \frac{\nabla^2}{4m_N^2})\frac{e^{-m_\phi \overline{r}}}{\overline{r}}$$

and

$$A = (\tau_1 \cdot \tau_2)(g_\rho^2 + g_\rho'^2)\frac{e^{-m_\rho \overline{r}}}{\overline{r}} + (g_w^2 + g_w'^2)\frac{e^{-m_w \overline{r}}}{\overline{r}} + (g_\phi^2 + g_\phi'^2)\frac{e^{-m_\phi \overline{r}}}{\overline{r}} \tag{5.42}$$

where $g_\rho'^2$, $g_\omega'^2$, $g_\phi'^2$ are also coupling constants we will fit.

5.4 Modification Of Our Phase Shift Equations

The phase shift equation which we used in our calculation are different from Eq(4.17) and Eq(4.39) to Eq(4.41). Because of the small r behavior of our potentials, these phase shift equations leads to numerical instability if we use Eq(4.17) and Eq(4.39) to Eq(4.41) directly in our calculations, especially, for p states and the coupled states. For $^1S_0$ we never have any problems which other states may have. The forms of Eq(4.17) and Eq(4.39) to Eq(4.41) for different angular momentum states are different, because they have different angular momentum barrier terms. To solve our problems, we put all the angular momentum barrier terms in the potentials, and change all the phase shift equations to the form of S state-like phase shift equations. So our phase shift equations are in a much simpler form.

For spin singlet states, our phase shift equations become

$$\delta_l'(r) = -k^{-1} V_l(r) \sin^2[kr + \delta_l(r)] \tag{5.43}$$

this equation is similar to $^1S_0$ state phase equation(see Eq.(4.16)), but it works well for all the singlet states when the angular momentum barrier terms ($\frac{l(l+1)}{r^2}$) are included in $V_l(r)$.

$$V_l(r) = V(r) + \frac{l(l+1)}{r^2}$$

For spin triplet states, our phase shift equations become(see Eq(4.38))

$$\frac{d}{dr}\delta_i(r) = -\frac{2m}{k} < \mathbf{v}_i(r), \mathbf{V}(r)\mathbf{v}_i(r) > \sin^2(kr + \delta_i(r)), \qquad i = 1, 2, \tag{5.44}$$

$$\sin(\delta_1(r) - \delta_2(r))\frac{d}{dr}\varepsilon(r) = -\frac{2m}{k} < \mathbf{v}_1(r), \mathbf{V}(r)\mathbf{v}_2(r) > \cdot \sin(kr + \delta_1(r))\sin(kr + \delta_2(r)) \tag{5.45}$$

Because the nucleon-nucleon interaction are short range, we integrate our phase shift equations(for both the singlet and triplet states) to an distance(for example 6 fm) where the nucleon-nucleon potential become very weak. Then only the angular momentum barrier terms $\frac{l(l+1)}{r^2}$ dominate the potential $V_l(r)$, then we let our potential $V_l(r) = \frac{l(l+1)}{r^2}$, and integrate our phase shift equations from 6 fm to infinity to get our phase shift.

Because of the modification of our phase shift equations, we also need to modify our boundary conditions for phase shift equations. For the uncoupled singlet states $^1P_1$, $^1D_2$ and triplet states $^3P_0$, $^3P_1$, the modified boundary conditions are[38]

$$\delta_l'(0) = -\frac{l}{l+1}k.$$

This is implemented by an additional boundary conditions at $r = h$, so our boundary conditions for uncoupled singlet states $^1P_1$, $^1D_2$ and triplet states $^3P_0$, $^3P_1$ are

$$\delta_l(h) = -\frac{l}{l+1}kh \tag{5.46}$$

where h is stepsize in our calculation, $k = \sqrt{\frac{2\mu E}{\hbar^2}}$. So for P and D states, the new boundary conditions are $\delta_1(h) = -\frac{1}{2}kh$, $\delta_2(h) = -\frac{2}{3}kh$ respectively.

For the coupled $^3S_1$ and $^3D_1$ states, the way to get the boundary conditions is a little bit tricky. We start with Eq(4.27), note all the phase shift equations are S state-like, so that the differential equation for the matrix T becomes

$$T' = -\frac{1}{k}[\sin^2(kr)\Phi + \sin(kr)\cos(kr)(\Phi T + T\Phi) + \cos^2(kr)T\Phi T] \tag{5.47}$$

Here, we use Φ to stand for the potentials in matrix form. We try to use this equation to get the boundary conditions $\varepsilon(h)$, $\delta_-(h)$ and $\delta_+(h)$.

Assuming at small r

$$T(r) = T(0) + rT'(0)$$

where(see Eq(4.28))

$$T(0) = 0$$

so

$$T(h) = hT'(0) \tag{5.48}$$

at small r, we can approximate our Φ as(see Φ_{11} and Φ_{12} in Eq(3.207) and Φ_{21} and Φ_{22} in Eq(3.208). Here we let $\Phi_{12} = \Phi_{21} \to \frac{1}{2}(\Phi_{12} + \Phi_{21})$)

$$\Phi = \frac{1}{r^2}\begin{pmatrix} \eta_- & \eta_0 \\ \eta_0 & 6 + \eta_+ \end{pmatrix} \tag{5.49}$$

substitute Eq(5.48) and Eq(5.49) we can find

$$T(h) = T(0) + hT'(0) = h \begin{pmatrix} \alpha & \beta \\ \beta & -\frac{2}{3} + \gamma \end{pmatrix} \tag{5.50}$$

where

$$\alpha = -\eta_-,$$

$$\beta = -\frac{1}{3}\eta_0$$

$$\gamma = -\frac{\eta_+}{45}$$

then we can find $\varepsilon(h)$, $\tan\delta_-(h)$ and $\tan\delta_+(h)$ by diagonalizing matrix $T(h)$. The matrix to diagonalize matrix $T(h)$ is(see Eq(4.24))

$$\begin{pmatrix} \cos\varepsilon & -\sin\varepsilon \\ \sin\varepsilon & \cos\varepsilon \end{pmatrix}.$$

By doing this, we can obtain

$$\tan(2\varepsilon) = \frac{\frac{2}{3}\eta_0}{\eta_- - (\frac{2}{3} + \frac{\eta_+}{45})} \tag{5.51}$$

$$\tan\delta_-(h) = T_{11} = -\eta_- \cos^2\varepsilon - \frac{2}{3}\eta_0 \cos\varepsilon \sin\varepsilon - (\frac{2}{3} + \frac{\eta_+}{45}) \sin^2\varepsilon \tag{5.52}$$

$$\tan \delta_+(h) = T_{22} = -\eta_- \sin^2 \varepsilon + \frac{2}{3}\eta_0 \cos \varepsilon \sin \varepsilon - (\frac{2}{3} + \frac{\eta_+}{45}) \cos^2 \varepsilon \tag{5.53}$$

Now, we can go ahead to do our calculation.

Chapter 6

Results

It is our aim to determine if a semi-phenomenological description is adequate for the N-N potential by incorporating the meson exchange model and the two body Dirac equations of constraint dynamics. In contrast to the equations used in Gross's and other people's[12,40,41,42,43,45] semi-phenomenological fitting, the two body Dirac equations of constraint dynamics can be exactly reduced to local Schrödinger-like equations. This allows us to gain the physical insight into the nucleon-nucleon interactions without making any assumptions and approximations that are questionable. We try to test our models to find which one can give us a best fit to experimental phase shift data in the nucleon-nucleon scattering. We have performed a bunch of model tests(two of which are presented here).

The data set[46] which we used in our test consist of all pp and np nucleon-nucleon scattering phase shift data below $T_{Lab} = 350\ MeV$, all the data are published in physics journals between 1955 and 1992. All phase shifts and mixing parameters were determined accurately. The data can give very accurate predictions for all nucleon-nucleon scattering data at any angle and any energy below 350 MeV. Since the data did not include any constraints on the energy behavior of the phase shifts beyond 350 MeV, the results at the high end of energy range($T_{Lab} \geq 325$ MeV) may be somewhat less reliable. Not all the angular momentum state experimental phase shifts are listed with errors . For the data that have errors, the errors are all very small, this mean all the data were measured very accurately. In our fitting, we use experimental phase shift data for NN scattering in the singlet states ${}^1S_0$, ${}^1P_1$, ${}^1D_2$ and triplet states ${}^3P_0$, ${}^3P_1$, ${}^3S_1$, ${}^3D_1$. Because we intend to use our fitting results to predicate the result in *pp* scattering, so we did not put the *pp* scattering data of singlet states${}^1S_0$, ${}^1D_2$ and triplet states ${}^3P_0$, ${}^3P_1$ into our fitting(There is no *pp* scattering in ${}^1P_1$, ${}^3S_1$ and ${}^3D_1$ states because of the consideration of Pauli principle).

Since we are not at the phase to get the perfect fitting and we are still at the stage to test our models, to expedite our calculation, we do not use the errors which the data provide with us, we use one degree as our errors in our fitting. We still have a lot of room to improve our model.

We use the 7 angular momentum states in our fit. There are 11 data points for every angular momentum state, in the energy range from 1 to 350 MeV, so the total numbers of data points in our fitting are 77. To determine the free coupling constant(and the sigma mass m_σ) in our potentials, we have to perform a best fit to the experimentally measured phase shift data. The coupling constant are generally searched by minimizing the quantity χ^2 . The definition of our χ^2 is

$$\chi^2 = \sum_i \{\frac{\delta_i^{th} - \delta_i^{\exp}}{\Delta\delta_i}\}^2 \tag{6.1}$$

where the δ_i^{th} is theoretical phase shifts, the $\delta_i^{\exp}$ is experimental phase shifts and we let $\Delta\delta_i = 1$ degree.

We have tried several methods to minimize our χ^2. The gradient method, grid method and Monte Carlo simulations. Our χ^2 drops very quickly at beginning if we search by the gradient method, then it always hits some local minima and can not jump out. Obviously, the grid method should lead us to the global minimum. The problem is that if we want find our parameters we must let the mesh small enough, but the calculation time become unbearably long, if we choose larger mesh, we miss the parameters which we are looking for.

We found that the Monte Carlo method can solve above dilemma. We set a reasonable range for all the parameters which we want to fit and generate all our fitting parameters randomly. Initially, the calculation time also very long for this method, but it can leads us to a rough area where our fitting parameters are located in. Then we shrink the range for all our fitting parameters and do our calculation again, our calculation time then being greatly reduced. By repeating several time in the same way, we can finally find the parameters.

To expedite our calculations further, we put restrictions on ${}^1S_0$ and ${}^3S_1$ states. After every set of parameters is generated randomly, we first test it on the ${}^1S_0$ state at 1 MeV. For ${}^1S_0$ state, if

$$| \delta_i^{th} - \delta_i^{\exp} | > 0.2 | \delta_i^{\exp} | \tag{6.2}$$

we let the computer jump out this loop and generate another set of parameters and test again until a set of parameters pass this restriction, then we test it on the ${}^3S_1$ states at 1 MeV with the same restriction. We only calculate δ_i^{th} at higher energy if a set of parameters pass these two restrictions. Otherwise, to continue to calculate δ_i^{th} at higher energy just waste time and is meaningless. Our code can run at least 50 times faster by this two restrictions. After we shrink our parameter ranges 2 or 3 times, all of our parameters are confined in a small region. At this time, we may change our restriction to

$$| \delta_i^{th} - \delta_i^{\exp} | > 0.15 | \delta_i^{\exp} | \tag{6.3}$$

and put restriction on ${}^1P_1$ states or any other states to let our code run more efficiently.

Using the method we discussed in this chapter, we try several different models to fit the phase shift experimental data of seven different angular momentum states which include the singlet states ${}^1S_0$, ${}^1P_1$, ${}^1D_2$ and triplet states ${}^3P_0$, ${}^3P_1$, ${}^3S_1$, ${}^3D_1$. Two models which we discussed in chapter 5 can give us fairly good fit to the experimental data. The parameters which we obtained for model 1 are listed in table 6.1, and for model 2 are listed in table 6.2. For the features of mesons in table 6.1 and 6.2, please refer to table 2.1 and Eq(5.28) to Eq(5.30).

Table 6.1: Parameters From Fitting Experimental Data(Model 1).

	η	η'	σ	ρ	ω	π	a_0
g^2	2.25	4.80	47.9	11.6	16.5	13.3	0.13
$r_0(\times 10^{-3})$	2.843	2.843	2.843	2.843	2.843	0.645	2.843
	ϕ	f_0	ρ'	ω'	ϕ'	m_σ	
g^2	5.64	19.9	0.34	20.6	3.10	724.1	
$r_0(\times 10^{-3})$	2.843	2.843	2.843	2.843	2.843	——	

Table 6.2: Parameters From Fitting Experimental Data(Model 2).

	η	η'	σ	ρ	ω	π	a_0
g^2	0.88	1.70	54.7	2.58	18.3	13.6	10.5
$r_0(\times 10^{-3})$	1.336	1.264	3.180	6.640	2.627	1.717	9.282
	ϕ	f_0	ρ'	ω'	ϕ'	m_σ	
g^2	9.12	33.5	5.11	28.6	12.1	694.3	
$r_0(\times 10^{-3})$	11.45	4.447	6.640	2.627	11.45	——	

Model 1 The theoretical phase shifts which we calculated by using the parameters for model 1 and the experimental phase shifts for all the seven states are listed in table 6.3. We use parameters given above results to predict the phase shift of *pp* scattering. Our prediction for the four *pp* scattering states which include singlet states $^1S_0$, $^1D_2$ and triplet states $^3P_0$, $^3P_1$ are listed in table 6.4. The results for *np* scattering are also figured from figure 6.1 to figure 6.7 and for *pp* scattering are also figured from figure 6.8 to figure 6.11.

Model 2 The theoretical phase shifts which we calculated by using the parameters for model 2 and the experimental phase shifts for all the seven states are listed in table 6.5. We also use the parameters for model 2 to predict the phase shift of *pp* scattering.

The prediction for the those four pp scattering states are listed in table 6.6. The results of model 2 for np scattering are also figured from figure 6.12 to figure 6.18 and for pp scattering are also figured from figure 6.19 to figure 6.22.

Our results show that our fit is still not perfect, especially, for the higher angular momentum states at higher energy. Probably, we have not set our model in the right way. This means we have not put our angular momentum and energy related terms in our potentials in the way they should be. The two body Dirac equations are relativistic wave equations, they should give the correct results for higher angular momentum states and at higher energy. So before we can obtain a perfect fit we still have a lot of work to do in building our model. The other reason that may cause this problem is that we did not include tensor and pseudovector interactions in our potentials. Maybe we should also include some other elements that we have not realized yet.

Our results in pp scattering show that if we obtain a very good fit in np scattering our predicted results in pp scattering will also very good. This means that it is unnecessary to include pp scattering in the our fit, we may use the parameters obtained in np scattering to predict the results in pp scattering. This indicates that our results are promising and our two body Dirac equations of constraint dynamics and the meson exchange model are suitable to construct semi-phenomenological potentials of the N-N scattering.

In contrast to Gross's semi-phenomenological potentials[45], we use nine mesons(see section 5.2 of chapter 5) in our fit, but Gross just use four mesons in one set of fit and six mesons in another set of fit. He just ignores the other mesons which, as a matter of fact, exist in the nucleon nucleon interactions.

Although Gross's curves are better than our curves, our theories are different. His relativistic wave equation can be reduced Schrödinger-like equation only in nonrelativistic limit. He made some assumptions and ignored a lot of hard handled nonlocal potential terms in his derivations. In contrast to Gross's relativistic and semi-phenomenological wave equations, two body Dirac equations of constraint dynamics can be exactly reduced to local Schrödinger-like equations. This allows us to gain the physical insight into the nucleon nucleon interactions without making any assumptions that are questionable. Furthermore, two body Dirac equations of constraint dynamics has been successfully applied and tested in numerical calculation in atomic physics and particle physics. So we believe that the two body Dirac equations of constraint dynamics should also give a good results in phase shift analysis of nuclear physics.

Table 6.3: np Scattering Phase Shift Of $^1S_0$, $^1P_1$, $^1D_2$, $^3P_0$, $^3P_1$, $^3S_1$ And $^3D_1$ States(Model 1).

Energy	$^1S_0$		$^1P_1$		$^1D_2$		$^3P_0$	
(MeV)	Exp.	The.	Exp.	The.	Exp.	The.	Exp.	The.
1	62.07	59.96	-0.187	-0.359	0.00	0.00	0.18	0.00
5	63.63	63.48	-1.487	-1.169	0.04	0.00	1.63	1.55
10	59.96	60.40	-3.039	-2.870	0.16	0.05	3.65	3.57
25	50.90	51.95	-6.311	-6.641	0.68	0.52	8.13	8.72
50	40.54	41.65	-9.670	-10.23	1.73	1.13	10.70	11.62
100	26.78	26.64	-14.52	-13.49	3.90	2.00	8.460	10.17
150	16.94	15.18	-18.65	-15.26	5.79	2.51	3.690	5.688
200	8.940	5.615	-22.18	-16.49	7.29	2.91	-1.44	0.66
250	1.960	-2.719	-25.13	-17.60	8.53	3.11	-6.51	-4.38
300	-4.460	-10.16	-27.58	-18.63	9.69	3.55	-11.47	-9.206
350	-10.59	-16.94	-29.66	-19.68	10.96	3.311	-16.39	-13.81
Energy	$^3P_1$		$^3S_1$		$^3D_1$		ε	
(MeV)	Exp.	The.	Exp.	The.	Exp.	The.	Exp.	The.
1	-0.11	-0.33	147.747	142.692	-0.005	0.719	0.105	0.287
5	-0.94	-0.88	118.178	112.670	-0.183	-0.176	0.672	1.224
10	-2.06	-2.26	102.611	98.215	-0.677	-0.256	1.159	1.951
25	-4.88	-5.70	80.63	78.38	-2.799	-2.910	1.793	2.587
50	-8.25	-10.18	62.77	62.00	-6.433	-6.947	2.109	2.495
100	-13.24	-16.66	43.23	43.18	-12.23	-13.94	2.420	3.013
150	-17.46	-22.12	30.72	30.64	-16.48	-19.35	2.750	3.562
200	-21.30	-26.98	21.22	20.95	-19.71	-23.78	3.130	4.489
250	-24.84	-31.46	13.39	12.95	-22.21	-27.62	3.560	5.682
300	-28.07	-35.67	6.600	6.127	-24.14	-31.01	4.030	6.982
350	-30.97	-39.58	0.502	0.171	-25.57	-34.15	4.570	8.536

Table 6.4: *pp* Scattering Phase Shift Of $^1S_0$, $^1D_2$, $^3P_0$ And $^3P_1$ States(Model 1).

Energy	$^1S_0$		$^1D_2$		$^3P_0$		$^3P_1$	
MeV	Exp.	The.	Exp.	The.	Exp.	The.	Exp.	The.
1	32.68	51.95	0.001	-0.091	0.134	0.381	-0.081	-1.215
5	54.83	55.47	0.043	-0.183	1.582	0.954	-0.902	-2.536
10	55.22	54.45	0.165	-0.270	3.729	1.773	-2.060	-3.864
25	48.67	47.64	0.696	-0.441	8.575	5.422	-4.932	-7.932
50	38.90	37.77	1.711	-0.504	11.47	9.766	-8.317	-13.15
100	24.97	23.63	3.790	0.511	9.450	7.862	-13.26	-18.45
150	14.75	12.37	5.606	1.141	4.740	3.812	-17.43	-24.42
200	6.550	3.024	7.058	2.407	–0.370	-1.178	-21.25	-28.50
250	-0.31	-5.15	8.270	2.994	-5.430	-6.193	-24.77	-33.26
300	-6.15	-12.55	9.420	3.136	-10.39	-10.98	-27.99	-37.63
350	-11.13	-19.27	10.69	2.902	-15.30	-15.42	-30.89	-41.13

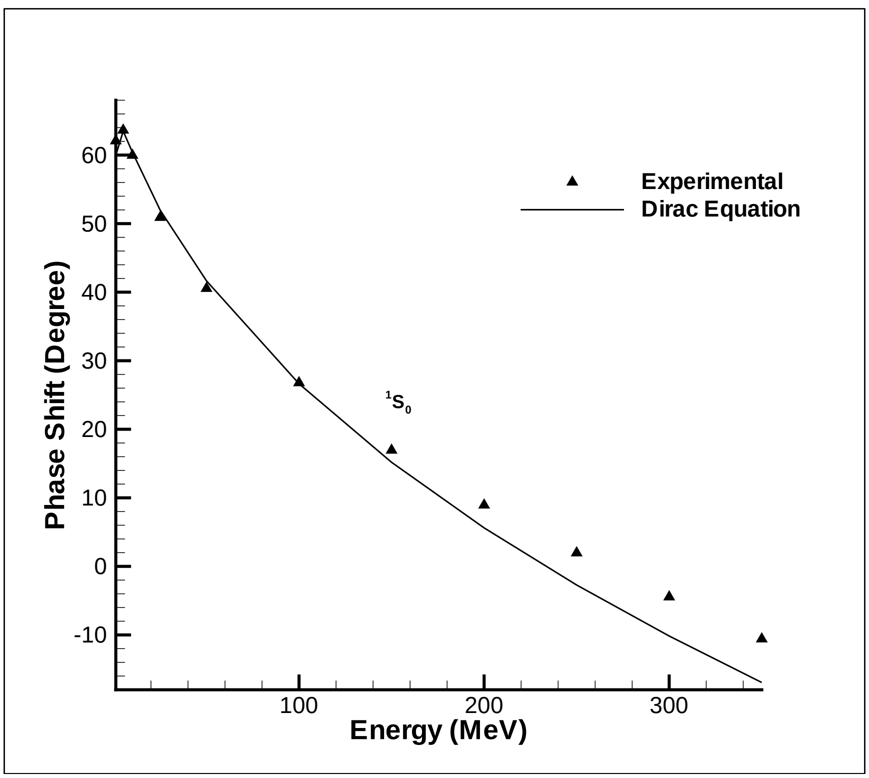

Figure 6.1: np Scattering Phase Shift of $^1S_0$ State(Model 1).

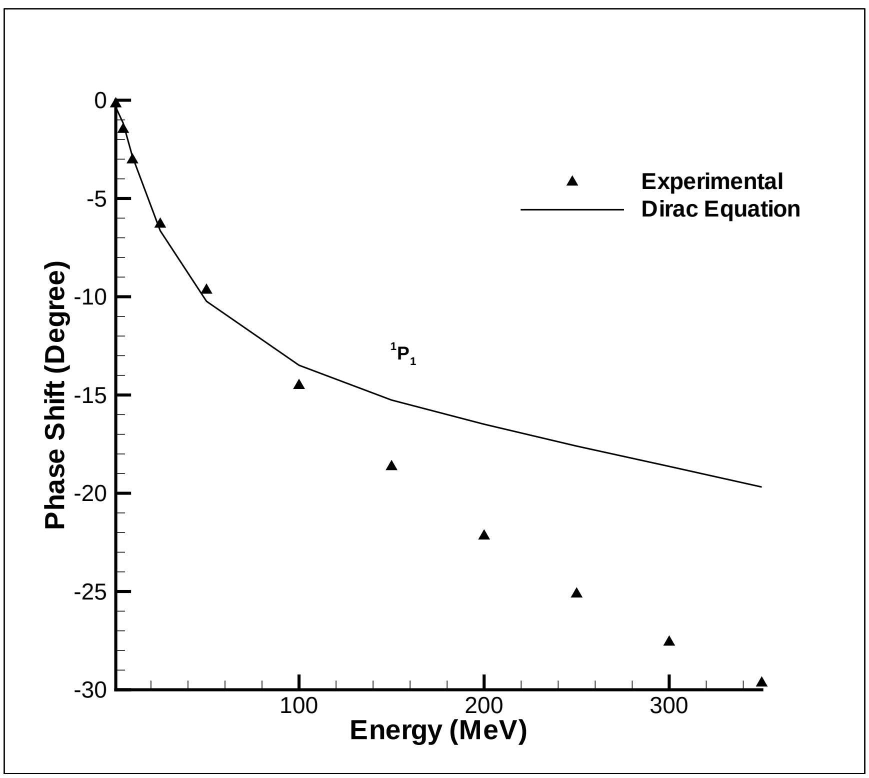

Figure 6.2: np Scattering Phase Shift of $^1P_1$ State(Model 1).

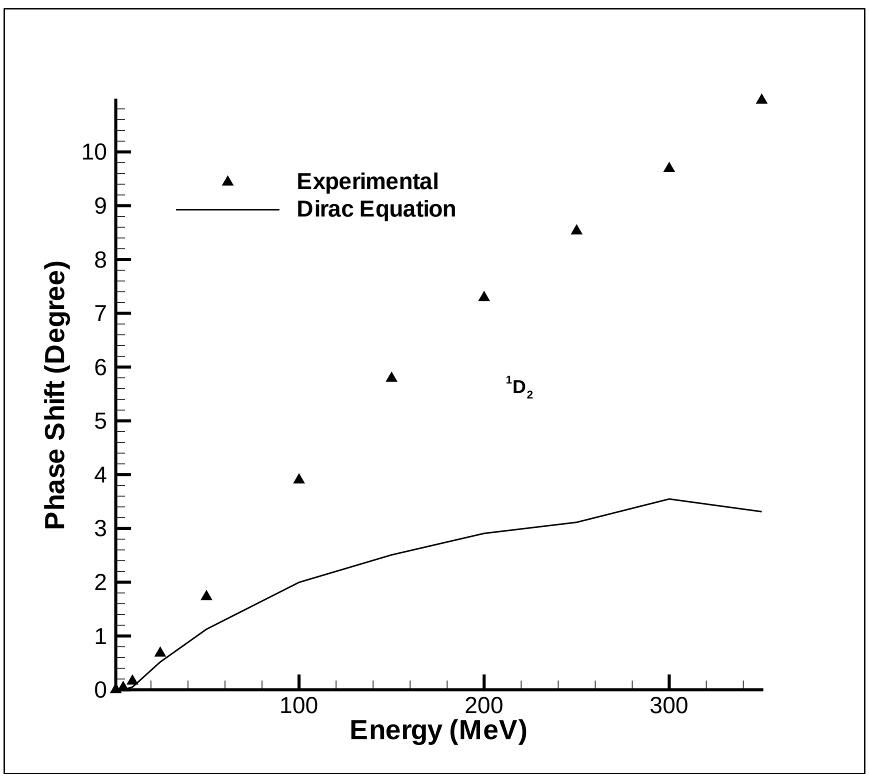

Figure 6.3: np Scattering Phase Shift of $^1D_2$ State(Model 1).

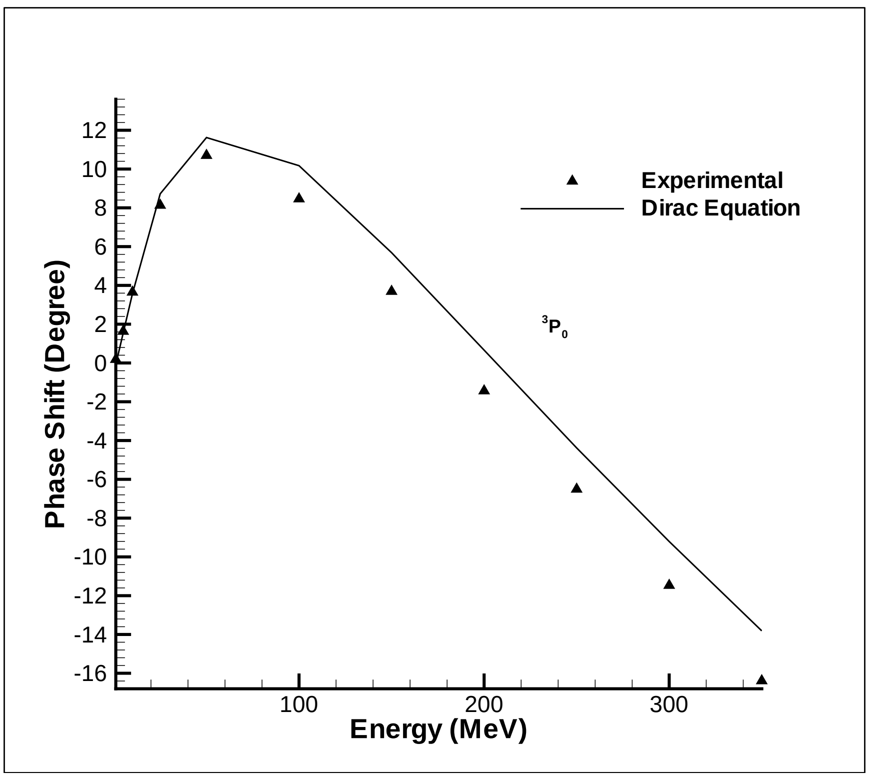

Figure 6.4: np Scattering Phase Shift of $^3P_0$ State(Model 1).

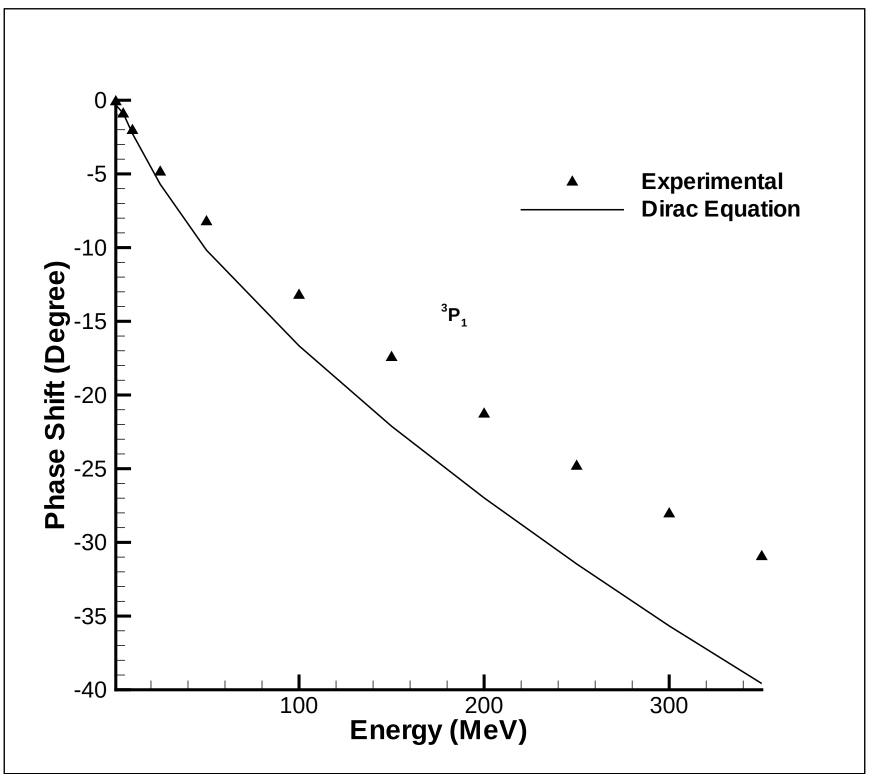

Figure 6.5: np Scattering Phase Shift of $^3P_1$ State(Model 1).

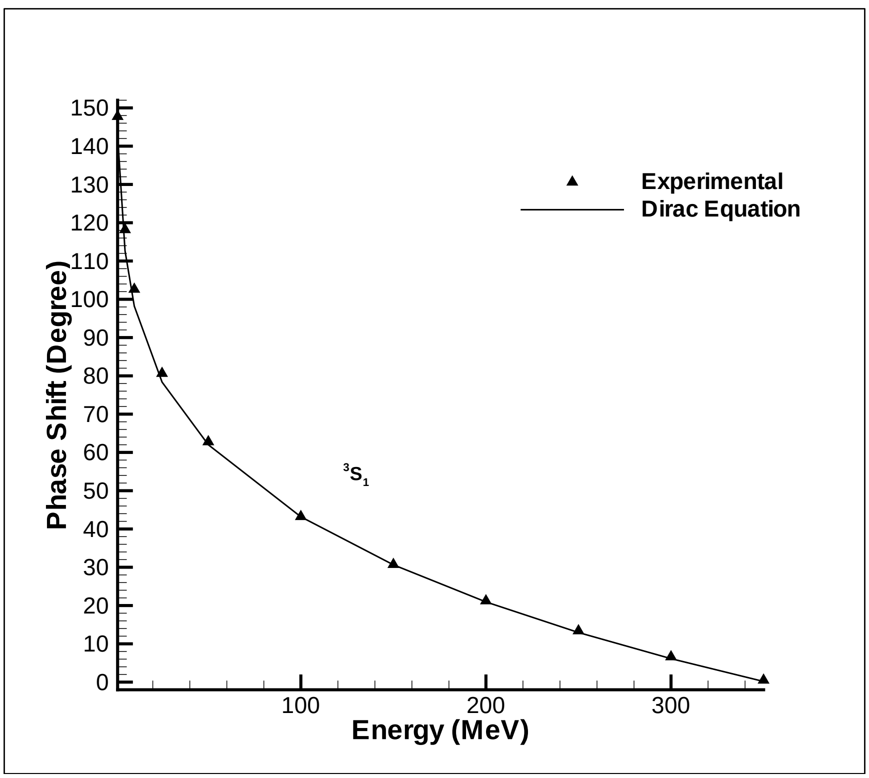

Figure 6.6: np Scattering Phase Shift of $^3S_1$ State(Model 1).

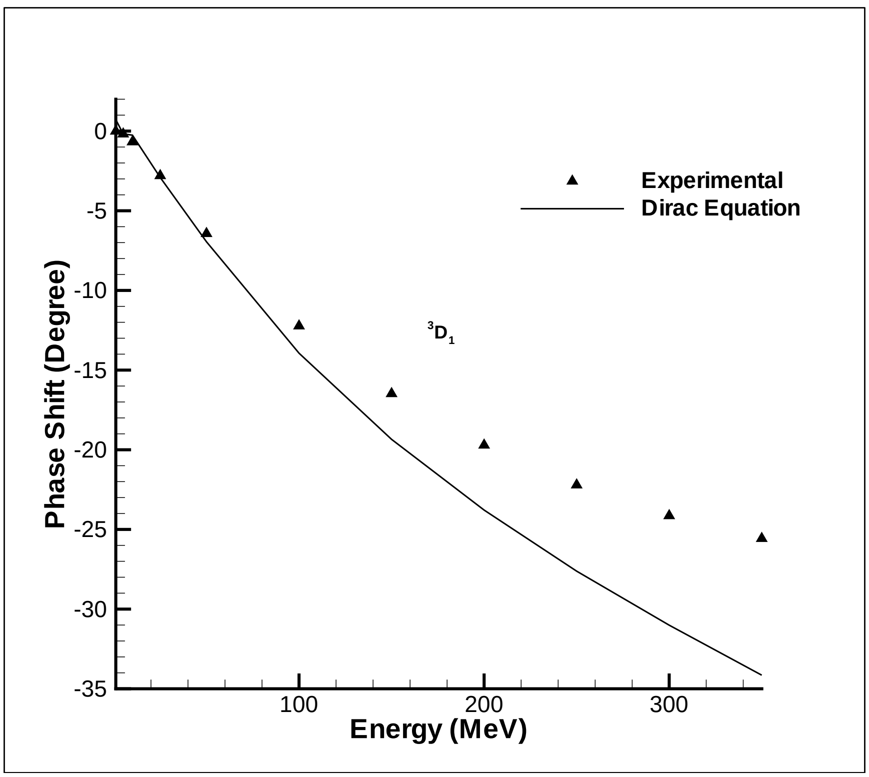

Figure 6.7: np Scattering Phase Shift of $^3D_1$ State(Model 1).

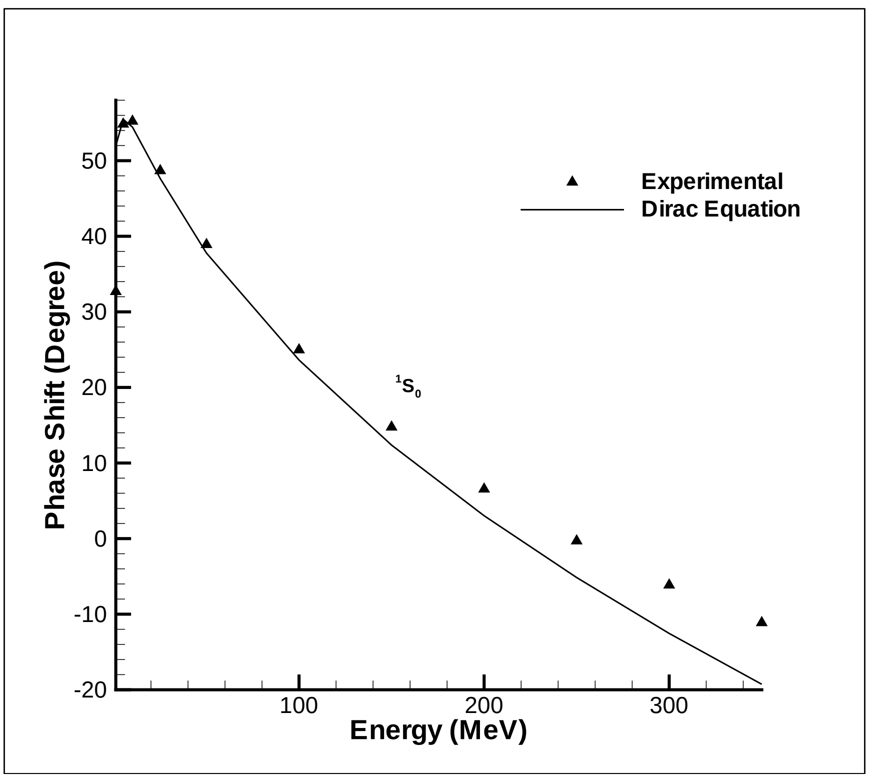

Figure 6.8: pp Scattering Phase Shift of $^1S_0$ State(Model 1).

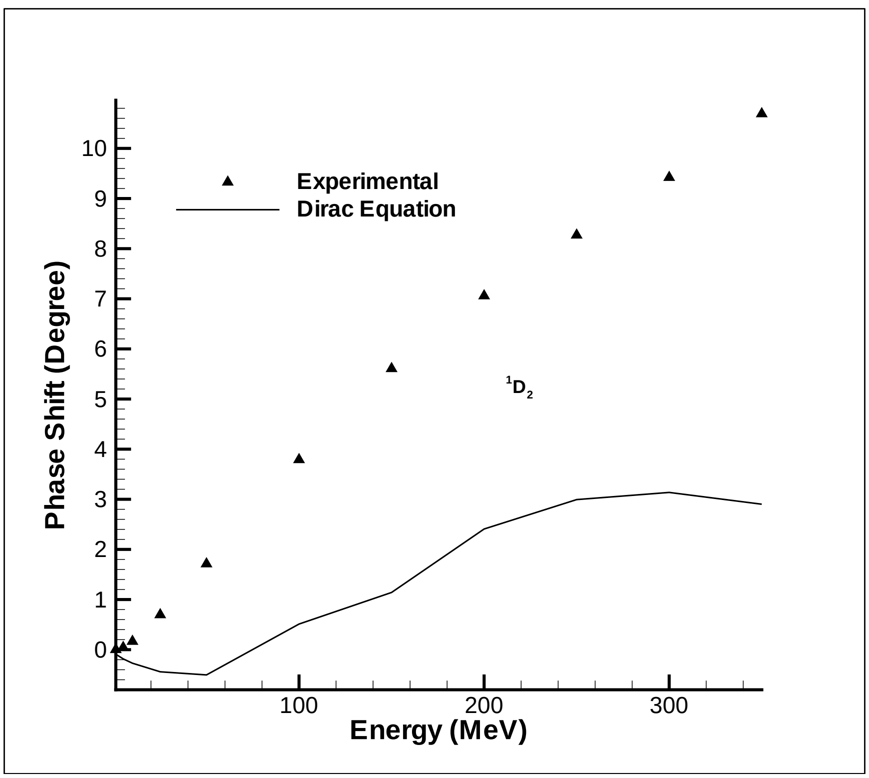

Figure 6.9: pp Scattering Phase Shift of $^1D_2$ State(Model 1).

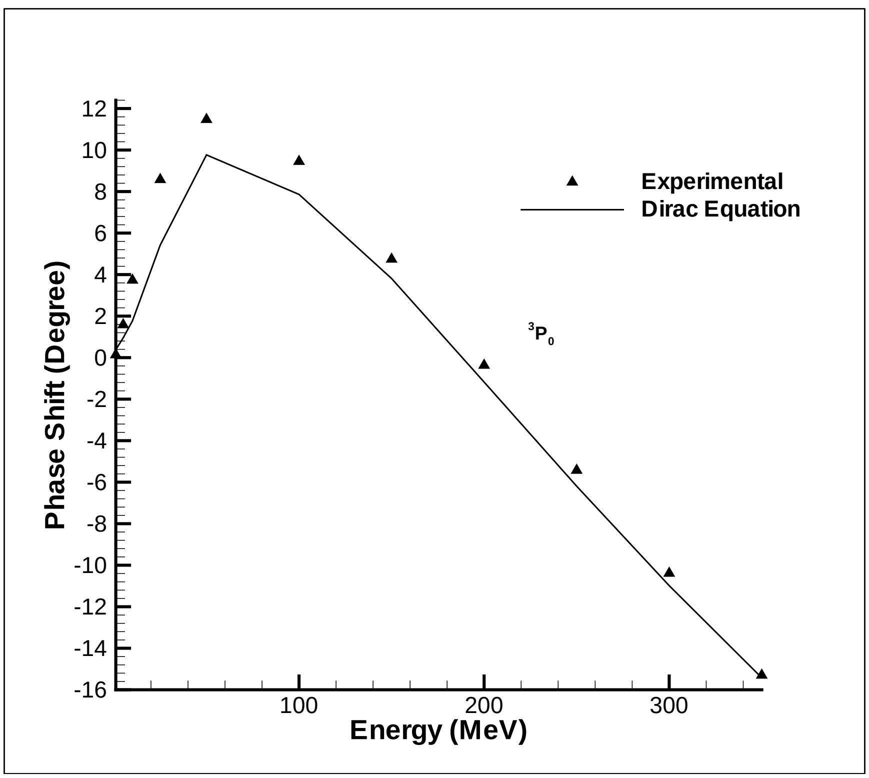

Figure 6.10: pp Scattering Phase Shift of $^3P_0$ State(Model 1).

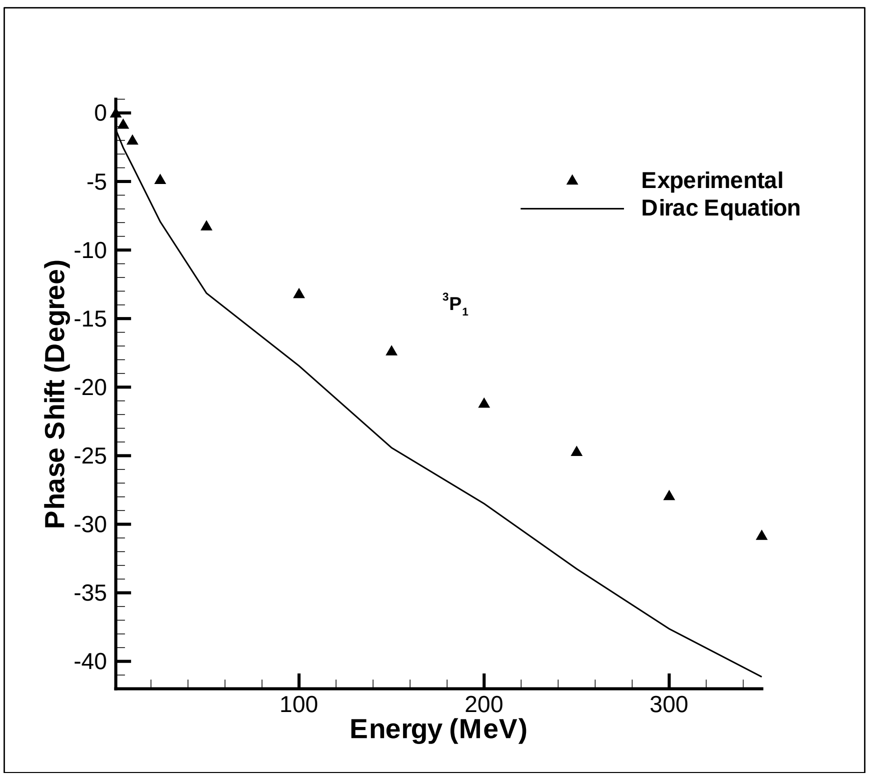

Figure 6.11: pp Scattering Phase Shift of $^3P_1$ State(Model 1).

Table 6.5: np Scattering Phase Shift Of $^1S_0$, $^1P_1$, $^1D_2$, $^3P_0$, $^3P_1$, $^3S_1$ And $^3D_1$ States(Model 2).

Energy	$^1S_0$		$^1P_1$		$^1D_2$		$^3P_0$	
(MeV)	Exp.	The.	Exp.	The.	Exp.	The.	Exp.	The.
1	62.07	60.60	-0.187	-0.358	0.00	0.02	0.18	0.00
5	63.63	63.50	-1.487	-1.163	0.04	0.15	1.63	1.61
10	59.96	60.20	-3.039	-2.857	0.16	0.39	3.65	3.74
25	50.90	51.44	-6.311	-6.629	0.68	0.40	8.13	9.28
50	40.54	40.91	-9.670	-10.36	1.73	1.37	10.70	12.69
100	26.78	25.86	-14.52	-14.44	3.90	2.42	8.460	11.74
150	16.94	14.62	-18.65	-17.55	5.79	3.62	3.690	7.399
200	8.940	5.435	-22.18	-20.37	7.29	4.55	-1.44	2.36
250	1.960	-2.428	-25.13	-23.15	8.53	5.24	-6.51	-2.78
300	-4.460	-9.330	-27.58	-25.87	9.69	5.34	-11.47	-7.746
350	-10.59	-15.52	-29.66	-28.54	10.96	5.30	-16.39	-12.52
Energy	$^3P_1$		$^3S_1$		$^3D_1$		ε	
(MeV)	Exp.	The.	Exp.	The.	Exp.	The.	Exp.	The.
1	-0.11	-0.32	147.747	144.797	-0.005	0.719	0.105	0.264
5	-0.94	-0.81	118.178	115.232	-0.183	-0.172	0.672	1.106
10	-2.06	-2.08	102.611	100.668	-0.677	-0.239	1.159	1.723
25	-4.88	-5.07	80.63	80.66	-2.799	-2.834	1.793	2.099
50	-8.25	-8.68	62.77	64.30	-6.433	-6.798	2.109	1.708
100	-13.24	-13.55	43.23	45.68	-12.23	-13.77	2.420	1.663
150	-17.46	-17.74	30.72	33.35	-16.48	-19.34	2.750	1.541
200	-21.30	-21.67	21.22	23.80	-19.71	-24.11	3.130	1.648
250	-24.84	-25.47	13.39	15.90	-22.21	-28.38	3.560	1.834
300	-28.07	-29.14	6.600	9.099	-24.14	-32.29	4.030	1.965
350	-30.97	-32.67	0.502	3.095	-25.57	-36.01	4.570	2.147

Table 6.6: pp Scattering Phase Shift Of $^1S_0$, $^1D_2$, $^3P_0$ And $^3P_1$ States(Model 2).

Energy	$^1S_0$		$^1D_2$		$^3P_0$		$^3P_1$	
MeV	Exp.	The.	Exp.	The.	Exp.	The.	Exp.	The.
1	32.68	52.40	0.001	-0.116	0.134	0.417	-0.081	-1.172
5	54.83	55.48	0.043	-0.232	1.582	1.042	-0.902	-2.434
10	55.22	54.24	0.165	-0.327	3.729	1.934	-2.060	-3.682
25	48.67	47.13	0.696	-0.524	8.575	5.943	-4.932	-7.355
50	38.90	37.04	1.711	-0.505	11.47	10.88	-8.317	-11.57
100	24.97	22.85	3.790	0.994	9.450	9.417	-13.26	-15.41
150	14.75	11.82	5.606	2.036	4.740	5.543	-17.43	-19.97
200	6.550	2.845	7.058	3.211	–0.370	0.495	-21.25	-23.23
250	-0.31	-4.86	8.270	3.648	-5.430	-4.589	-24.77	-27.28
300	-6.15	-11.72	9.420	3.956	-10.39	-9.516	-27.99	-31.05
350	-11.13	-17.85	10.69	4.014	-15.30	-14.13	-30.89	-34.22

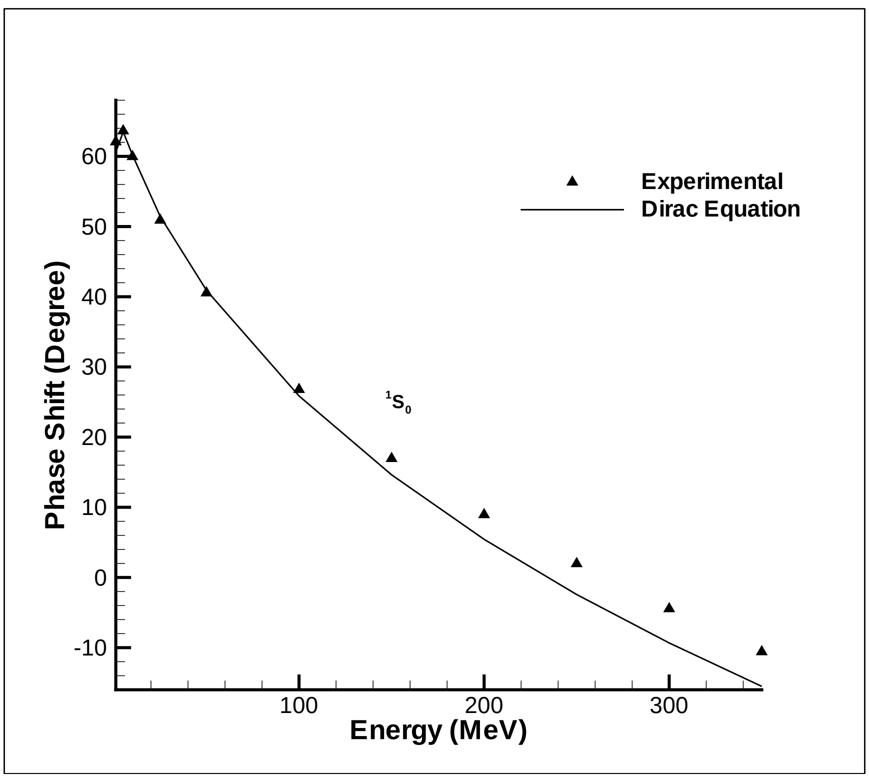

Figure 6.12: np Scattering Phase Shift of $^1S_0$ State(Model 2).

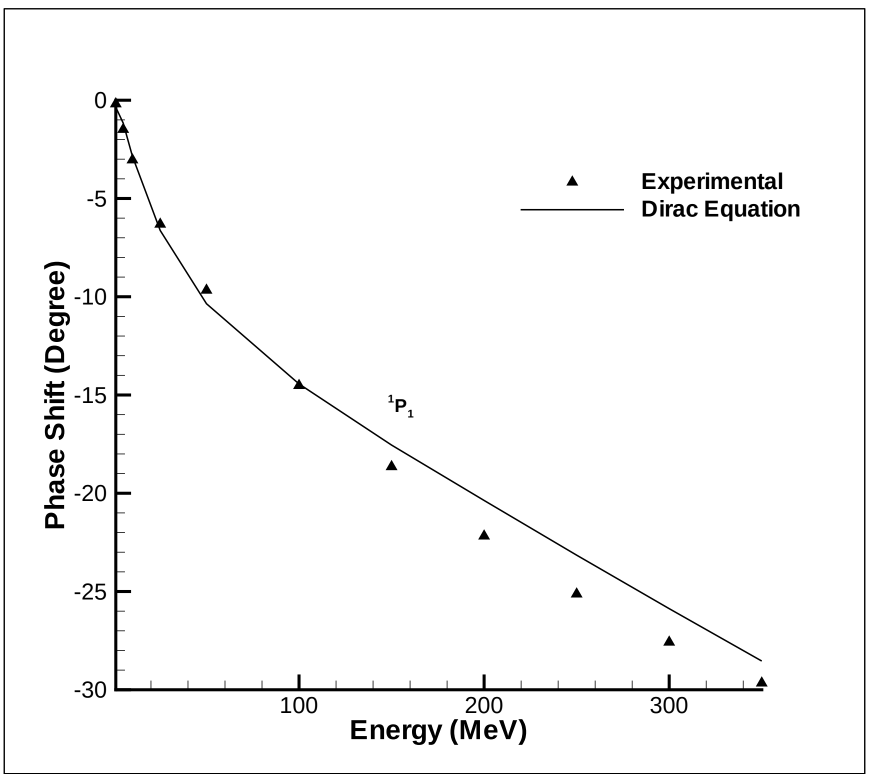

Figure 6.13: np Scattering Phase Shift of $^1P_1$ State(Model 2).

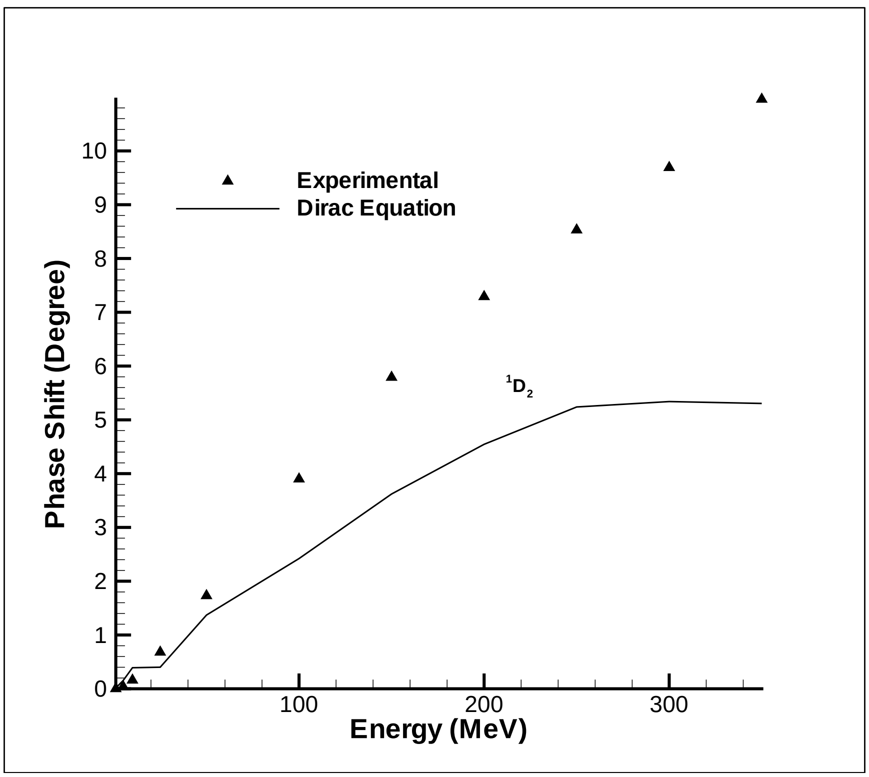

Figure 6.14: np Scattering Phase Shift of $^1D_2$ State(Model 2).

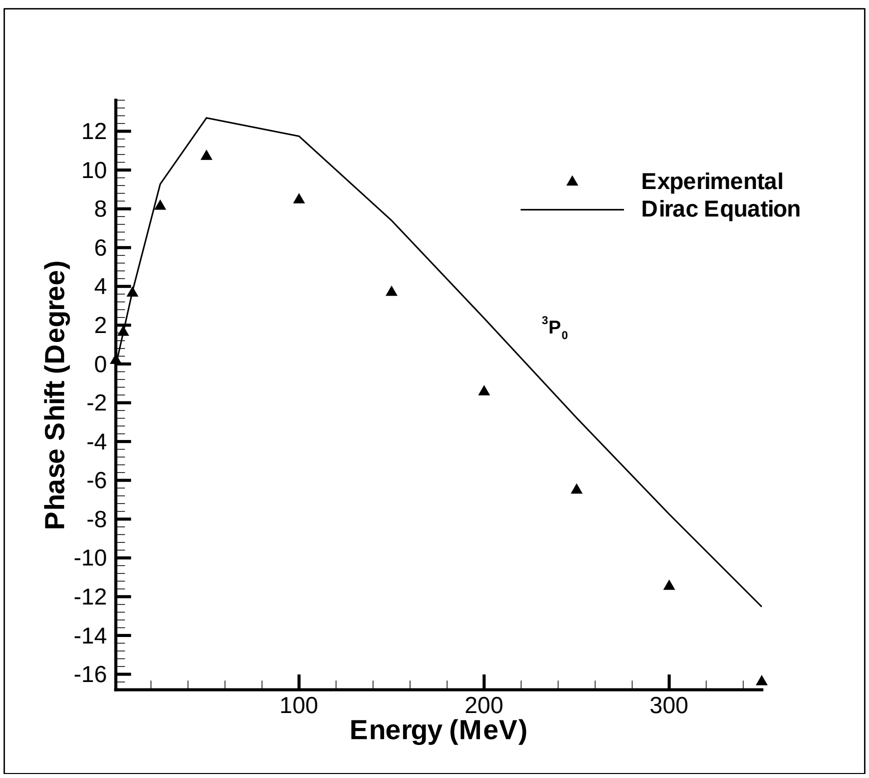

Figure 6.15: np Scattering Phase Shift of $^3P_0$ State(Model 2).

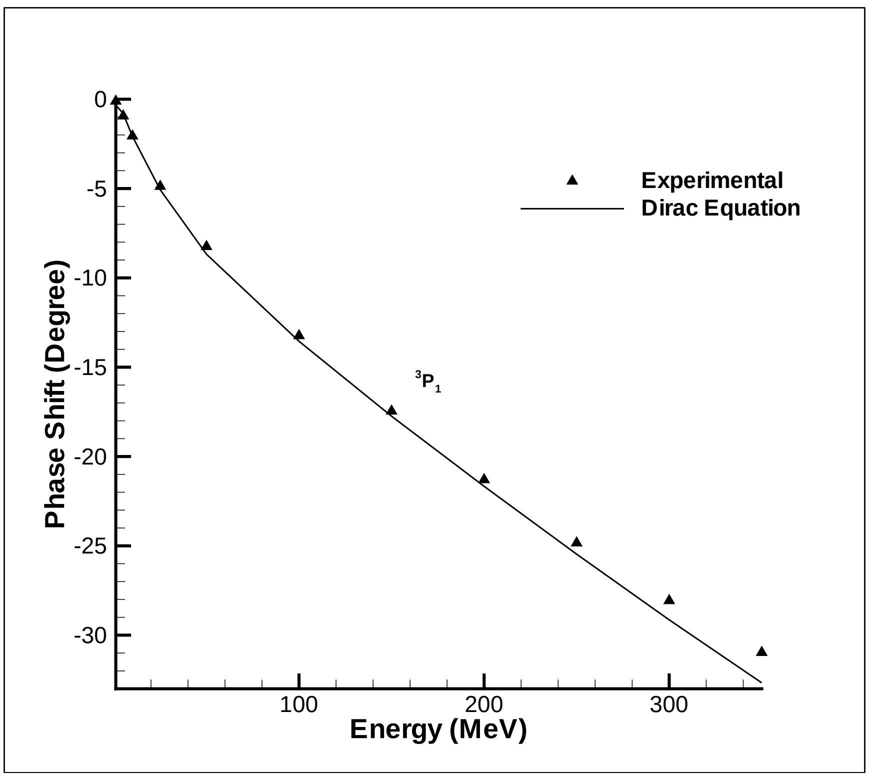

Figure 6.16: np Scattering Phase Shift of $^3P_1$ State(Model 2).

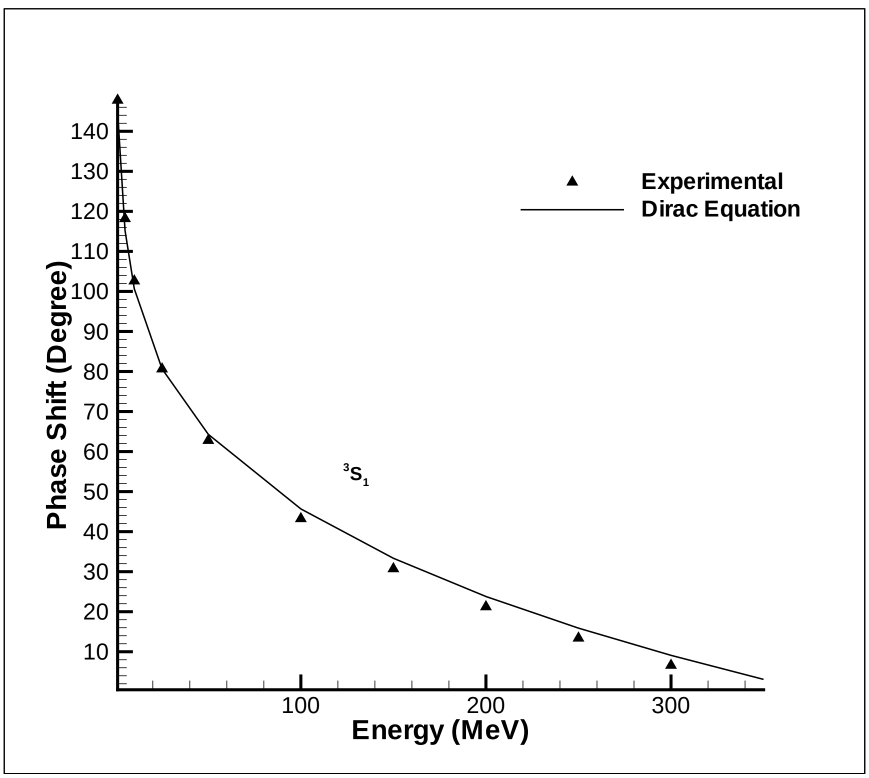

Figure 6.17: np Scattering Phase Shift of $^3S_1$ State(Model 2).

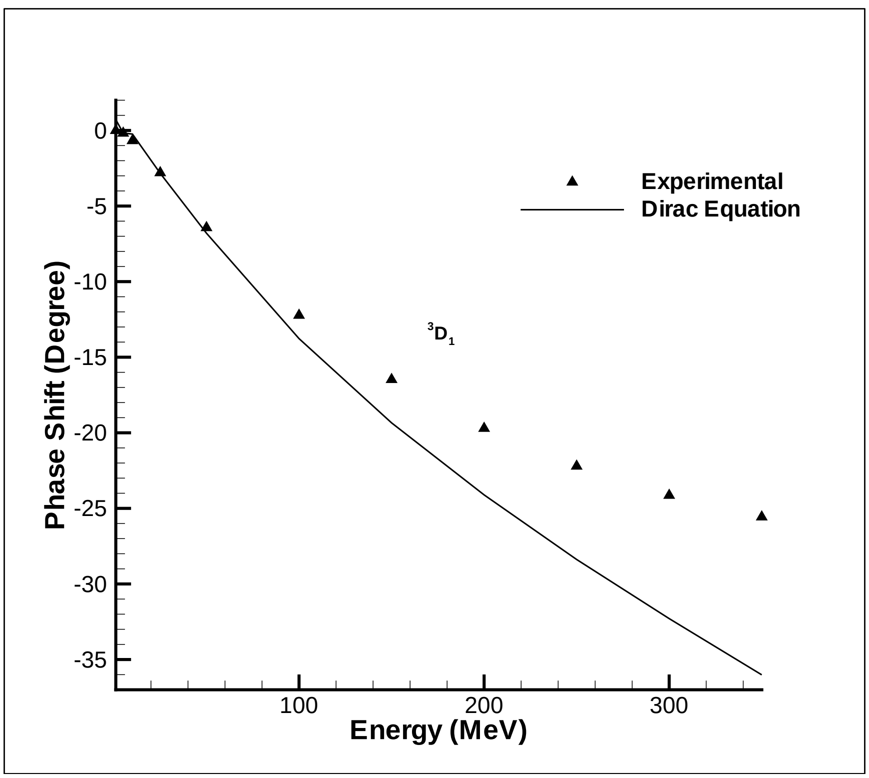

Figure 6.18: np Scattering Phase Shift of $^3D_1$ State(Model 2).

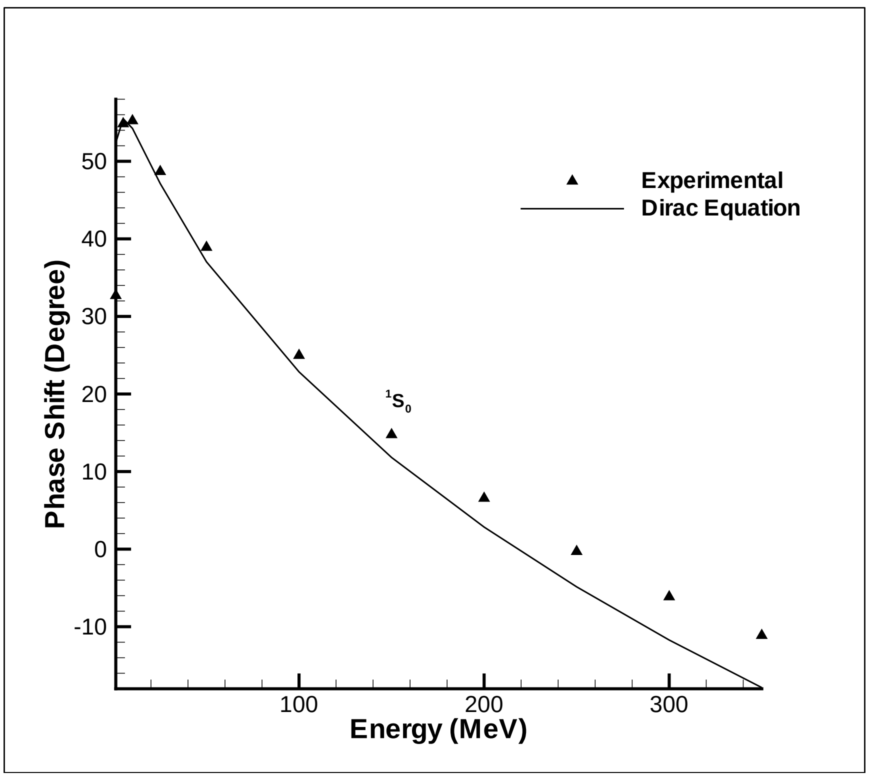

Figure 6.19: pp Scattering Phase Shift of $^1S_0$ State(Model 2).

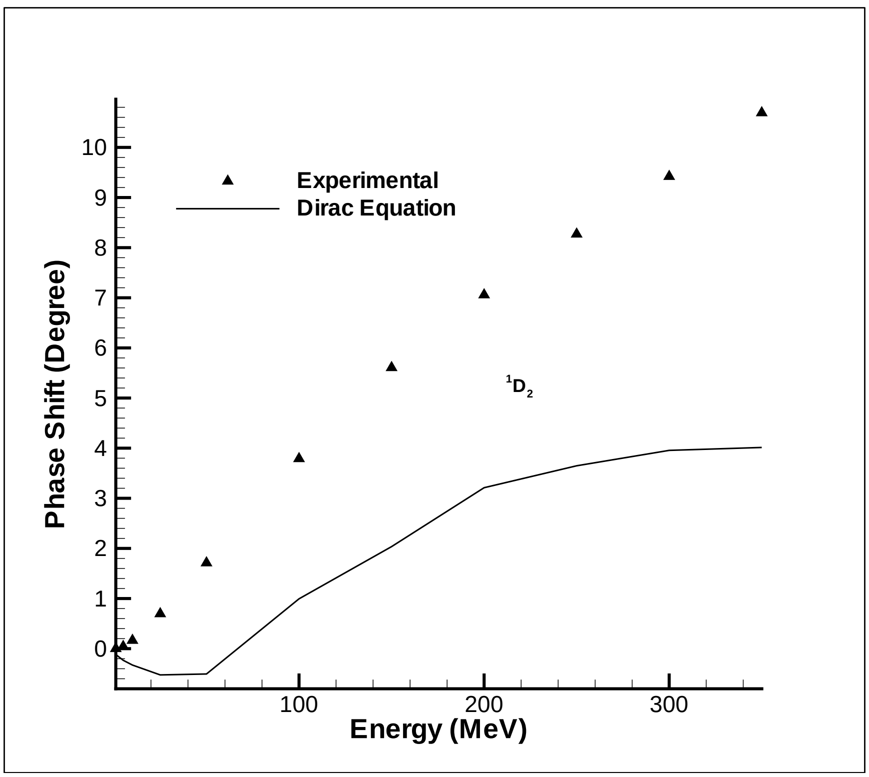

Figure 6.20: pp Scattering Phase Shift of $^1D_2$ State(Model 2).

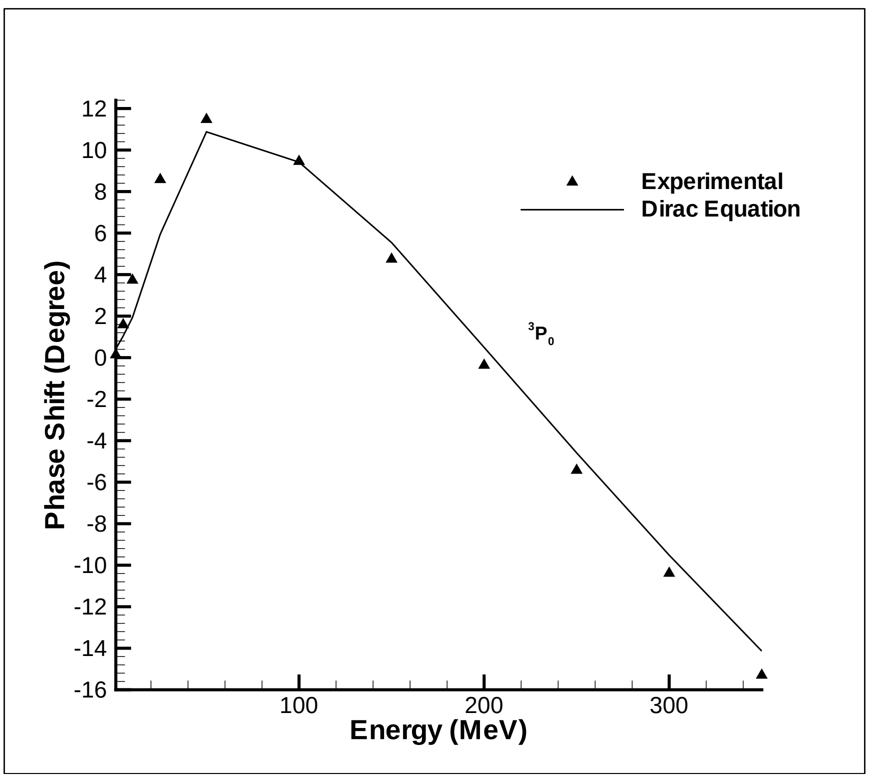

Figure 6.21: pp Scattering Phase Shift of $^3P_0$ State(Model 2).

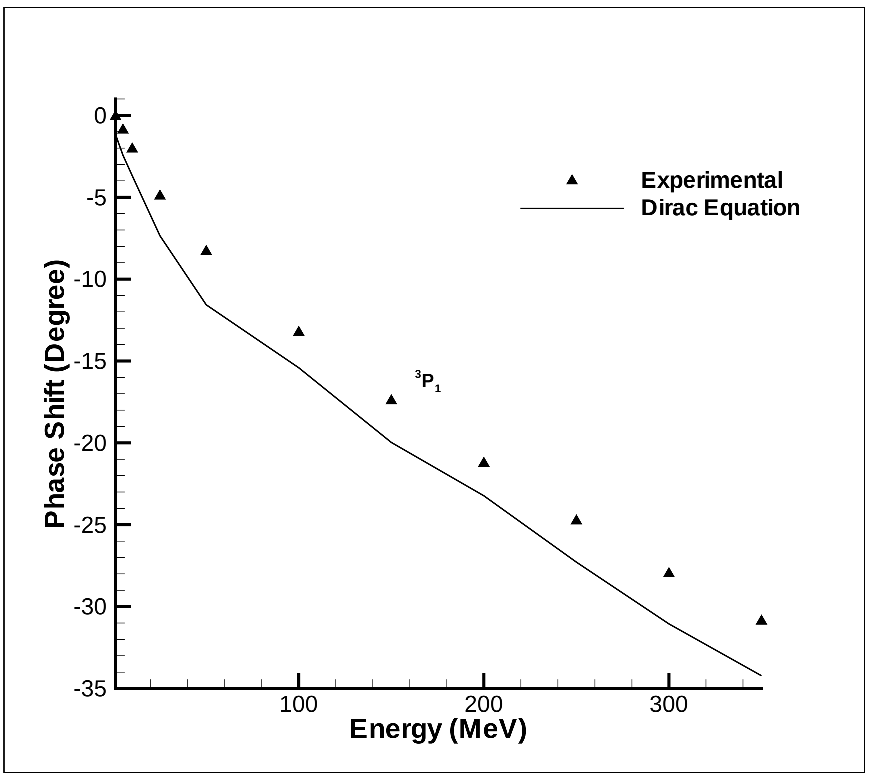

Figure 6.22: pp Scattering Phase Shift of $^3P_1$ State(Model 2).

Chapter 7

Summary and Suggestions

7.1 Summary

In this dissertation, we have reduced the two-body Dirac equations to coupled Schrödinger-like equations. This work had been done in Long's paper[31], but we use a completely different derivation method. After we made several minor corrections in Long's derivation, our equations get an agreement with his results although our derivation procedure is completely different. So we are confident of our derivation procedure and our results. This is very important for my dissertation because this is just the first step of my dissertation. If I made a error at this step, all my following work becomes a lemon.

After we reduce the two-body Dirac equations to a coupled Schrödinger-like equations, these Schrödinger-like equations have first derivative terms which are inherent in the two body Dirac equations. This is a difference with the standard Schrödinger-like equation. Before we can apply the techniques which have been already developed for the Schrödinger-like system in nonrelativistic quantum mechanics, we must get rid of these first derivative terms . This work is an important and crucial analytical part of this dissertation. For the uncoupled states, it is pretty straightforward. For the coupled states, the radial eigenvalue equations is in a matrix form and the process in getting rid of the first derivative term is very complicated. We were not successful in getting rid of first derivative terms for coupled states starting from the coupled radial eigenvalue equations. However, we developed a different approach that works for both the uncoupled and coupled states simultaneously.

We have tested several models by using the variable phase methods. Because of the small r behaviors of our potential if we use Eq(4.17) and Eq(4.39) to Eq(4.41) directly in our calculations, especially for p states and the coupled states, these phase shift equations lead to numerical instability. We really experienced some hard time to solve this problem. But finally, we found we never have such problem with $^1S_0$ state. The forms of Eq(4.17) and Eq(4.39) to Eq(4.41) for different angular momentum states are different, because they have different angular momentum barrier terms. To solve our problems, we put all the angular momentum barrier terms in the potentials, and change all the phase shift equations to the form of S state-like phase shift equations. So the phase shift equations which we used in our calculation are different from Eq(4.17) and Eq(4.39) to Eq(4.41) which are the usually phase shift equations used by other people, our phase shift equations are in a much simpler form.

After several models and several methods to minimize our χ^2 tested, we found two models(see chapter 5) which can lead us to a fairly good fit to the experimental phase shift data. This means that our work have showed a promising result. Here is a brief summary of the important equation in this dissertation.

Our Master Equation

$$\{\mathbf{p}^2 + \frac{2g' \sinh^2(K)}{r} - g'F' - 2h'K' - 4h'\frac{\cosh(K)\sinh(K)}{r}$$

$$-F'' - F'^2 - K'^2 - \frac{2}{r}F' - 4\frac{\sinh^2(K)}{r^2}$$

$$+\vec{L}\cdot(\sigma_1+\sigma_2)[\frac{g'}{2r} + \frac{g'\sinh^2(K)}{r} - \frac{2\sinh^2(K)}{r^2} - 2h'\frac{\cosh(K)\sinh(K)}{r}]$$

$$+ \quad \sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}\vec{L}\cdot(\sigma_1+\sigma_2)(2h'\frac{\sinh^2(K)}{r} + 2\frac{\sinh(K)\cosh(K)}{r^2} + \frac{h'}{r} - \frac{g'\sinh(K)\cosh(K)}{r})$$

$$+ \quad \sigma_1\cdot\sigma_2[k + \frac{g'\cosh(K)\sinh(K)}{r} + \frac{g'\sinh^2(K)}{r} - 2h'\frac{\cosh(K)\sinh(K)}{r}$$

$$-2h'\frac{\sinh^2(K)}{r} - 2\frac{\cosh(K)\sinh(K)}{r^2} - 2\frac{\sinh^2(K)}{r^2}] +$$

$$\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}[j - \frac{3g'\cosh(K)\sinh(K)}{r} - \frac{g'\sinh^2 K}{r} - g'K' - 2h'F' + \frac{2h'\cosh K\sinh K}{r}$$

$$
+6h'\frac{\sinh^2(K)}{r} - (2F'K' + K'') - \frac{2}{r}K' + 6\frac{\cosh(K)\sinh(K)}{r^2} + 2\frac{\sinh^2(K)}{r^2}] + m\}\psi
$$

$$
= \mathcal{B}^2 e^{-2\mathcal{G}}\psi
$$

see Eq(3.206) for the expression for g', h', K, K' and F'.

This is our master equation used in our phase shift analysis for nucleon nucleon scattering. It is a coupled Schrödinger-like equation derived from two body Dirac equations without making any assumption and approximation(beyond those assumed in chapter 5). All of our radial wave equations for any specific angular momentum state are obtained from this equation.

Effective Interactions We use nine mesons in our fit. We summarize the meson-nucleon interactions by writing the quantum field theory Lagrangian of their effective interactions

$$
\begin{aligned}
L_I = \; & g_\sigma\overline{\psi}\psi\sigma + g_{f_0}\overline{\psi}\psi f_0 + g_{a_0}\overline{\psi}\psi a_0 \\
& + g_\rho\overline{\psi}\gamma^\mu\overline{\tau}\psi\overrightarrow{\rho}_\mu + g_\omega\overline{\psi}\gamma^\mu\psi\omega_\mu + g_\phi\overline{\psi}\gamma^\mu\psi\phi_\mu \\
& + g_\pi\overline{\psi}\gamma^5\overline{\tau}\psi\overrightarrow{\pi} + g_\eta\overline{\psi}\gamma^5\psi\eta + g_{\eta'}\overline{\psi}\gamma^5\psi\eta'
\end{aligned} \tag{7.1}
$$

where ψ represent the nucleon field, σ, f_0, a_0, $\overrightarrow{\rho}$, ... represent the meson fields.

Variable Phase Methods The phase equations which we use in our calculation are different from the usual phase shift equations used by other people. Our phase shift equations are all in the S state-like form. For spin singlet states, our phase shift equations is in the form

$$\delta_l'(r) = -k^{-1} V_l(r) \sin^2[kr + \delta_l(r)]$$

this equation is similar to $^1S_0$ state phase equation, but it works well for all the spin singlet states when the angular momentum barrier terms ($\frac{l(l+1)}{r^2}$) are included in $V_l(r)$.

For spin triplet states, our phase shift equations are in the form

$$\frac{d}{dr}\delta_i(r) = -\frac{2m}{k} < \mathbf{v}_i(r), \mathbf{V}(r)\mathbf{v}_i(r) > \sin^2(kr + \delta_i(r)), \qquad i = 1, 2,$$

$$\sin(\delta_1(r) - \delta_2(r))\frac{d}{dr}\varepsilon(r) = -\frac{2m}{k} < \mathbf{v}_1(r), \mathbf{V}(r)\mathbf{v}_2(r) > \cdot \sin(kr + \delta_1(r)) \sin(kr + \delta_2(r))$$

again, the angular momentum barrier terms ($\frac{l(l+1)}{r^2}$) are included in $V_l(r)$.

Fitting the Phase Shift Experimental Data Several models have been tested by using the variable phase methods, two models can lead us to a fairly good fit to the experimental phase shift data. We use the parameters which gives the good fit to predict the phase shift for the *pp* scattering, we also get a good prediction for the *pp* scattering based on the parameters we obtained. This means that our work have showed a promising result. The following are some suggestions to improve our work in the future.

7.2 Suggestions For Future Work

Other Model Test More model testing is absolutely necessary in the future. By model we mean a way in which the perturbative interaction that arise from Eq(7.1)are put into the nonperturbative form we need for L, C and $\mathcal{G}$. During our fitting, we found that our final result are very sensitive to the model we chose. Some models give very bad fit, others give good fit. Changing the way to modify the interactions and the way of mesons enter into the two body Dirac equations can always produce new opportunity to improve our fit.

Include Tensor Interactions We just have included scalar, pseudoscalar and vector interactions in our potentials through the invariant forms like L, C and $\mathcal{G}$. Treating two body Dirac equations with tensor interactions of the vector meson may greatly improve our fit. These tensor interacting were discussed in chapter 5(see Eq(5.30)) and correspond to non-minimal coupling of spin one-half particle not present in QED but which can not be ruled out in massive vector meson-nucleon interactions. The corresponding field theory interaction is

$$\Delta L_I = g'_\rho \overline{\psi} \sigma^{\mu\nu} \overline{\tau} \psi \overrightarrow{\rho}_{\mu\nu} + g'_\omega \overline{\psi} \sigma^{\mu\nu} \psi \omega_{\mu\nu} + g'_\phi \overline{\psi} \sigma^{\mu\nu} \psi \phi_{\mu\nu}$$

and would correspond to relaxing the free field equation assumption made in Eq(5.38)).

Include Pseudovector Interactions Another option is to allow the pseudoscalar mesons(π, η, and η') to interact with the nucleon not only by the pseudoscalar interaction(as in Eq(7.1)) but also by the way of the pseudovector interactions as below

$$\Delta L_I = g'_\pi \overline{\psi} \gamma^\mu \gamma^5 \overline{\tau} \psi \partial_\mu \overrightarrow{\pi} + g'_\eta \overline{\psi} \gamma^\mu \gamma^5 \psi \partial_\mu \eta + g'_{\eta'} \overline{\psi} \gamma^\mu \gamma^5 \psi \partial_\mu \eta'$$

Include Full Massive Spin-One Propagator We have ignored a portion of the massive spin-one propagator in our fit since it is zero for particle on the mass shell. To include this portion of massive spin-one propagator we would change the vector propagator as below

$$\frac{\eta^{\mu\nu}}{q^2 + m_\rho^2 - i\varepsilon} \longrightarrow \frac{\eta^{\mu\nu} + \frac{q^\mu q^\nu}{m_\rho^2}}{q^2 + m_\rho^2 - i\varepsilon}$$

Among all the four suggestions, the first one is easy to do and have more chance to improve our fit. The last three suggestions would involve corresponding additions to the interaction that appear in the two body Dirac equations. In particular, we would have to modify our master equation which was derived from Long's equation. This means there is still a lot of analytical and numerical work to do to implement these three suggestions.

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Appendix

Appendix A

Riccatic-Bessel Functions

Definition of the Riccati-Bessel Functions

$$\hat{j}_l(x) = (\frac{\pi x}{2})^{\frac{1}{2}} J_{l+\frac{1}{2}}(x).$$

$$\hat{n}_l(x) = (-1)^{l+1} (\frac{\pi x}{2})^{\frac{1}{2}} J_{-(l+\frac{1}{2})}(x).$$

Here $J_p(x)$ is the standard Bessel function.

The First Riccati-Bessel Functions

$$\hat{j}_0(x) = \sin x,$$

$$\hat{j}_1(x) = -\cos x + \frac{\sin x}{x},$$

$$\hat{j}_2(x) = \sin x(\frac{3}{x^2} - 1) - \frac{3}{x}\cos x;$$

$$\hat{n}_0(x) = -\cos x,$$

$$\hat{n}_1(x) = -\sin x - \frac{\cos x}{x},$$

$$\hat{n}_2(x) = -\cos x(\frac{3}{x^2} - 1) - \frac{3}{x}\sin x.$$

Wronskian Relation

$$\hat{j}_l(x)\hat{n}_l'(x) - \hat{j}_l'(x)\hat{n}_l(x) = 1.$$

Behavior Near the Origin

$$\hat{j}_l(x) = [x^{l+1}/(2l+1)!!][1 + O(x^2)]$$

$$\hat{n}_l(x) = [-x^{-l}(2l-1)!!][1 + O(x^2)]$$

Asymptotic Behavior

$$\hat{j}_l(x) \xrightarrow{r\to\infty} \sin(x - \frac{l\pi}{2}) \qquad (x \gg l),$$

$$\hat{n}_l(x) \xrightarrow{r\to\infty} -\cos(x - \frac{l\pi}{2}) \qquad (x \gg l).$$

Parity

$$\hat{j}_l(-x) = (-1)^{l+1}\hat{j}_l(x)$$

$$\hat{n}_l(-x) = (-1)^l\hat{n}_l(x)$$

Amplitude and Phase of the Riccati-Bessel Functions

$$\hat{j}_l(x) = \hat{D}_l(x) \sin \hat{\delta}_l(x),$$

$$\hat{n}_l(x) = -\hat{D}_l(x) \cos \hat{\delta}_l(x),$$

$$\hat{D}_l(x) = [\hat{j}_l^2(x) + \hat{n}_l^2(x)]^{\frac{1}{2}},$$

$$\hat{\delta}_l(x) = -\tan^{-1}[\frac{\hat{j}_l(x)}{\hat{n}_l(x)}].$$

Appendix B

Reduction To Radial Form

B.1 Reduction To Radial Form

Our Eq.(3.173) is

$$h[E_1[\sigma_1 \cdot \mathbf{p} - i\sigma_2 \cdot (\mathbf{d} + \mathbf{k}\sigma_1 \cdot \sigma_2)]hF_1[\sigma_1 \cdot \mathbf{p} - i\sigma_2 \cdot (\mathbf{z} + \mathbf{k}\sigma_1 \cdot \sigma_2)]\phi_+ \qquad (a)$$

$$+h[M_1[\sigma_1 \cdot \mathbf{p} - i\sigma_2 \cdot (\mathbf{o} + \mathbf{k}\sigma_1 \cdot \sigma_2)]hF_3[\sigma_1 \cdot \mathbf{p} - i\sigma_2 \cdot (\mathbf{z} + \mathbf{k}\sigma_1 \cdot \sigma_2)]\phi_+ \qquad (b)$$

$$-h[E_1[\sigma_1 \cdot \mathbf{p} - i\sigma_2 \cdot (\mathbf{d} + \mathbf{k}\sigma_1 \cdot \sigma_2)]hF_2[\sigma_2 \cdot \mathbf{p} - i\sigma_1 \cdot (\mathbf{z} + \mathbf{k}\sigma_1 \cdot \sigma_2)]\phi_+ \qquad (c)$$

$$+h[M_1[\sigma_1 \cdot \mathbf{p} - i\sigma_2 \cdot (\mathbf{o} + \mathbf{k}\sigma_1 \cdot \sigma_2)]hF_4[\sigma_2 \cdot \mathbf{p} - i\sigma_1 \cdot (\mathbf{z} + \mathbf{k}\sigma_1 \cdot \sigma_2)]\phi_+ \qquad (d)$$

$$= \mathcal{B}^2\phi_+. \qquad \text{(B.1)}$$

For future reference we will refer to the four sets of terms on the left hand side as Eq.(B.1) (a),(b),(c),(d) term.

Substitute $\mathbf{d}$, h, F_1, $\mathbf{z}$, $\mathbf{k}$'s expressions to (a) term of Eq.(B.1), we obtain

$$
\begin{aligned}
(a) \ &= \mathrm{e}^{\mathcal{G}} E_1\{[\sigma_1\cdot\mathbf{p} - \frac{i}{2}\sigma_2\cdot\nabla(C+J+L) - \frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1 + i\sigma_1\times\sigma_2)] \\
&\times \mathrm{e}^{\mathcal{G}}\frac{M_2}{\mathcal{D}}[\sigma_1\cdot\mathbf{p} - \frac{i}{2}\sigma_2\cdot\nabla(-C+J-L) - \frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1 + i\sigma_1\times\sigma_2)]\}
\end{aligned}
\tag{B.2}
$$

using the commutation relation of $\mathbf{p}$ to move the second $\sigma_1\cdot\mathbf{p}$ term in above expression to the left, we can find the (a) term is

$$
\begin{aligned}
&\mathrm{e}^{\mathcal{G}} E_1\{[\mathrm{e}^{\mathcal{G}}\frac{M_2}{\mathcal{D}}[\sigma_1\cdot\mathbf{p} - \frac{i}{2}\sigma_2\cdot\nabla(-C+J-L) - \frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1 + i\sigma_1\times\sigma_2)](\sigma_1\cdot\mathbf{p}) \\
&+\frac{1}{i}\sigma_1\cdot\partial[\mathrm{e}^{\mathcal{G}}\frac{M_2}{\mathcal{D}}[\sigma_1\cdot\mathbf{p} - \frac{i}{2}\sigma_2\cdot\nabla(-C+J-L) - \frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1 + i\sigma_1\times\sigma_2)]] \\
&-\frac{i}{2}[\sigma_2\cdot\nabla(C+J+L) + \nabla\mathcal{G}\cdot(\sigma_1 + i\sigma_1\times\sigma_2)]\mathrm{e}^{\mathcal{G}}\frac{M_2}{\mathcal{D}}[\sigma_1\cdot\mathbf{p} - \frac{i}{2}\sigma_2\cdot\nabla(-C+J-L) \\
&-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1 + i\sigma_1\times\sigma_2)]\}
\end{aligned}
$$

simplify the first term in above expression by using

$$
(\sigma_1\cdot\mathbf{p})(\sigma_1\cdot\mathbf{p}) = \mathbf{p}^2,
$$

$$\sigma_1(\sigma_1 \cdot \mathbf{p}) = \mathbf{p} + \mathbf{i}(\sigma_\mathbf{1} \times \mathbf{p}),$$

$$(\sigma_1 \times \sigma_2)(\sigma_1 \cdot \mathbf{p}) = i(\sigma_1 \cdot \sigma_2)\mathbf{p} - i\sigma_1(\sigma_2 \cdot \mathbf{p}) - (\sigma_\mathbf{2} \times \mathbf{p}) \tag{B.3}$$

we can obtain

(a) term is

$$\mathrm{e}^{\mathcal{G}} E_1\{[\mathrm{e}^{\mathcal{G}} \frac{M_2}{\mathcal{D}}[\mathbf{p}^2 - \frac{i}{2}\sigma_2 \cdot \nabla(-C + J - L)(\sigma_1 \cdot \mathbf{p})$$

$$-\frac{i}{2}\nabla\mathcal{G} \cdot [(\mathbf{p} + i(\sigma_1 \times \mathbf{p}) - (\sigma_1 \cdot \sigma_2)\mathbf{p} + \sigma_1(\sigma_2 \cdot \mathbf{p}) - i(\sigma_2 \times \mathbf{p})]]$$

$$+\frac{1}{i}\sigma_1 \cdot \partial[\mathrm{e}^{\mathcal{G}} \frac{M_2}{\mathcal{D}}[\sigma_1 \cdot \mathbf{p} - \frac{i}{2}\sigma_2 \cdot \nabla(-C + J - L) - \frac{i}{2}\nabla\mathcal{G} \cdot (\sigma_1 + i\sigma_1 \times \sigma_2)]]$$

$$-\frac{i}{2}[\sigma_2 \cdot \nabla(C + J + L) + \nabla\mathcal{G} \cdot (\sigma_1 + i\sigma_1 \times \sigma_2)]\mathrm{e}^{\mathcal{G}} \frac{M_2}{\mathcal{D}}[\sigma_1 \cdot \mathbf{p}$$

$$-\frac{i}{2}\sigma_2 \cdot \nabla(-C + J - L) - \frac{i}{2}\nabla\mathcal{G} \cdot (\sigma_1 + i\sigma_1 \times \sigma_2)]\} \tag{B.4}$$

Substitute $\mathbf{o}$, h, F_1, $\mathbf{z}$, $\mathbf{k}$'s expressions to (b) term of Eq.(B.1), we obtain

$$(b)\ \ term = \mathrm{e}^{\mathcal{G}} M_1\{[\sigma_1 \cdot \mathbf{p} - \frac{i}{2}\sigma_2 \cdot \nabla(C - J - L) - \frac{i}{2}\nabla\mathcal{G} \cdot (\sigma_1 + i\sigma_1 \times \sigma_2)]$$

$$\times \mathrm{e}^{\mathcal{G}} \frac{E_2}{\mathcal{D}}[\sigma_1 \cdot \mathbf{p} - \frac{i}{2}\sigma_2 \cdot \nabla(-C + J - L) - \frac{i}{2}\nabla\mathcal{G} \cdot (\sigma_1 + i\sigma_1 \times \sigma_2)]\} \tag{B.5}$$

using the commutation relation of $\mathbf{p}$ to move the second $\sigma_1 \cdot \mathbf{p}$ term in above expression to the left, we can find the

(b) term is

$$e^{\mathcal{G}} M_1 \{[e^{\mathcal{G}} \frac{E_2}{\mathcal{D}} [\sigma_1 \cdot \mathbf{p} - \frac{i}{2} \sigma_2 \cdot \nabla(-C + J - L) - \frac{i}{2} \nabla \mathcal{G} \cdot (\sigma_1 + i \sigma_1 \times \sigma_2)](\sigma_1 \cdot \mathbf{p})$$

$$+ \frac{1}{i} \sigma_1 \cdot \partial [e^{\mathcal{G}} \frac{E_2}{\mathcal{D}} [\sigma_1 \cdot \mathbf{p} - \frac{i}{2} \sigma_2 \cdot \nabla(-C + J - L) - \frac{i}{2} \nabla \mathcal{G} \cdot (\sigma_1 + i \sigma_1 \times \sigma_2)]]$$

$$- \frac{i}{2} [\sigma_2 \cdot \nabla(C - J - L) + \nabla \mathcal{G} \cdot (\sigma_1 + i \sigma_1 \times \sigma_2)] e^{\mathcal{G}} \frac{E_2}{\mathcal{D}} [\sigma_1 \cdot \mathbf{p}$$

$$- \frac{i}{2} \sigma_2 \cdot \nabla(-C + J - L) - \frac{i}{2} \nabla \mathcal{G} \cdot (\sigma_1 + i \sigma_1 \times \sigma_2)]\}$$

simplify the first term in above expression by using Eq.(B.3)

(b) term is

$$e^{\mathcal{G}} M_1 \{ e^{\mathcal{G}} \frac{E_2}{\mathcal{D}} [\mathbf{p}^2 - \frac{i}{2} \sigma_2 \cdot \nabla(-C + J - L)(\sigma_1 \cdot \mathbf{p})$$

$$- \frac{i}{2} \nabla \mathcal{G} \cdot [(\mathbf{p} + i(\sigma_1 \times \mathbf{p}) - (\sigma_1 \cdot \sigma_2)\mathbf{p} + \sigma_1(\sigma_2 \cdot \mathbf{p}) - i(\sigma_2 \times \mathbf{p})]]$$

$$+ \frac{1}{i} \sigma_1 \cdot \partial [e^{\mathcal{G}} \frac{E_2}{\mathcal{D}} [\sigma_1 \cdot \mathbf{p} - \frac{i}{2} \sigma_2 \cdot \nabla(-C + J - L) - \frac{i}{2} \nabla \mathcal{G} \cdot (\sigma_1 + i \sigma_1 \times \sigma_2)]]$$

$$-\frac{i}{2}[\sigma_2\cdot\nabla(C-J-L)+\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]\mathrm{e}^{\mathcal{G}}\frac{E_2}{\mathcal{D}}[\sigma_1\cdot\mathbf{p}$$

$$-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]\} \tag{B.6}$$

Substitute $\mathbf{d}$, h, F_1, $\mathbf{z}$, $\mathbf{k}$'s expressions to (c) term of Eq.(B.1), we obtain

$$(c)\ \ term=-\mathrm{e}^{\mathcal{G}}E_1\{[\sigma_1\cdot\mathbf{p}-\frac{i}{2}\sigma_2\cdot\nabla(C+J+L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]$$

$$\times\mathrm{e}^{\mathcal{G}}\frac{M_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_2+i\sigma_2\times\sigma_1)]\} \tag{B.7}$$

using the commutation relation of $\mathbf{p}$ to move the second $\sigma_2\cdot\mathbf{p}$ term in above expression to the left, we can find the

(c) term is

$$-\mathrm{e}^{\mathcal{G}}E_1\{[\mathrm{e}^{\mathcal{G}}\frac{M_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_2+i\sigma_2\times\sigma_1)](\sigma_1\cdot\mathbf{p})$$

$$+\frac{1}{i}\sigma_1\cdot\partial[\mathrm{e}^{\mathcal{G}}\frac{M_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_2+i\sigma_2\times\sigma_1)]]$$

$$-\frac{i}{2}[\sigma_2\cdot\nabla(C+J+L)+\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]\mathrm{e}^{\mathcal{G}}\frac{M_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}$$

$$-\frac{i}{2}\sigma_1 \cdot \nabla(-C+J-L) - \frac{i}{2}\nabla\mathcal{G} \cdot (\sigma_2 + i\sigma_2 \times \sigma_1)]\}$$

simplify the first term in above expression by using Eq.(B.3)

(c) term is

$$-\mathrm{e}^{\mathcal{G}} E_1\{\mathrm{e}^{\mathcal{G}}\frac{M_1}{\mathcal{D}}[(\sigma_2 \cdot \mathbf{p})(\sigma_1 \cdot \mathbf{p}) - \frac{i}{2}\sigma_1 \cdot \nabla(-C+J-L)(\sigma_1 \cdot \mathbf{p})$$

$$-\frac{i}{2}\nabla\mathcal{G} \cdot [(\sigma_2(\sigma_1 \cdot \mathbf{p}) - (\sigma_1 \cdot \sigma_2)\mathbf{p} + \sigma_1(\sigma_2 \cdot \mathbf{p}) + i(\sigma_2 \times \mathbf{p})]]$$

$$+\frac{1}{i}\sigma_1 \cdot \partial[\mathrm{e}^{\mathcal{G}}\frac{M_1}{\mathcal{D}}[\sigma_2 \cdot \mathbf{p} - \frac{i}{2}\sigma_1 \cdot \nabla(-C+J-L) - \frac{i}{2}\nabla\mathcal{G} \cdot (\sigma_2 + i\sigma_2 \times \sigma_1)]]$$

$$-\frac{i}{2}[\sigma_2 \cdot \nabla(C+J+L) + \nabla\mathcal{G} \cdot (\sigma_1 + i\sigma_1 \times \sigma_2)]\mathrm{e}^{\mathcal{G}}\frac{M_1}{\mathcal{D}}[\sigma_2 \cdot \mathbf{p}$$

$$-\frac{i}{2}\sigma_1 \cdot \nabla(-C+J-L) - \frac{i}{2}\nabla\mathcal{G} \cdot (\sigma_2 + i\sigma_2 \times \sigma_1)]\} \tag{B.8}$$

Substitute $\mathbf{o}$, h, F_1, $\mathbf{z}$, $\mathbf{k}$'s expressions to (d) term of Eq.(B.1), we obtain

$$(d)\ \ term = \mathrm{e}^{\mathcal{G}} M_1\{[\sigma_1 \cdot \mathbf{p} - \frac{i}{2}\sigma_2 \cdot \nabla(C-J-L) - \frac{i}{2}\nabla\mathcal{G} \cdot (\sigma_1 + i\sigma_1 \times \sigma_2)]$$

$$\times\mathrm{e}^{\mathcal{G}}\frac{E_1}{\mathcal{D}}[\sigma_2 \cdot \mathbf{p} - \frac{i}{2}\sigma_1 \cdot \nabla(-C+J-L) - \frac{i}{2}\nabla\mathcal{G} \cdot (\sigma_2 + i\sigma_2 \times \sigma_1)]\} \tag{B.9}$$

using the commutation relation of $\mathbf{p}$ to move the second $\sigma_2 \cdot \mathbf{p}$ term in above expression to the left, we can find the (d) term is

$$e^{\mathcal{G}}M_1\{[e^{\mathcal{G}}\frac{E_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_2+i\sigma_2\times\sigma_1)](\sigma_1\cdot\mathbf{p})$$

$$+\frac{1}{i}\sigma_1\cdot\partial[e^{\mathcal{G}}\frac{E_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_2+i\sigma_2\times\sigma_1)]]$$

$$-\frac{i}{2}[\sigma_2\cdot\nabla(C-J-L)+\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]e^{\mathcal{G}}\frac{E_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}$$

$$-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_2+i\sigma_2\times\sigma_1)]\}$$

simplify the first term in above expression by using Eq.(B.3)

(d) term is

$$e^{\mathcal{G}}M_1\{e^{\mathcal{G}}\frac{E_1}{\mathcal{D}}[(\sigma_2\cdot\mathbf{p})(\sigma_1\cdot\mathbf{p})-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)(\sigma_1\cdot\mathbf{p})$$

$$-\frac{i}{2}\nabla\mathcal{G}\cdot[(\sigma_2(\sigma_1\cdot\mathbf{p})-(\sigma_1\cdot\sigma_2)\mathbf{p}+\sigma_1(\sigma_2\cdot\mathbf{p})+i(\sigma_2\times\mathbf{p})]]$$

$$+\frac{1}{i}\sigma_1\cdot\partial[e^{\mathcal{G}}\frac{E_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_2+i\sigma_2\times\sigma_1)]]$$

$$-\frac{i}{2}[\sigma_2\cdot\nabla(C-J-L)+\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]e^{\mathcal{G}}\frac{E_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}$$

$$-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_2+i\sigma_2\times\sigma_1)]\} \tag{B.10}$$

Combining all the (a),(b),(c),(d) term. Using $\mathcal{D}=E_1M_2+E_2M_1$, we can combine the first term of (a) term and (b) term, the first term of (c) term and (d) term cancelled each other, we also can combine the third term of (c) term and (d) term. Finally, we can get

(a) + (b) + (c) + (d) term=

$$\mathrm{e}^{\mathcal{G}}\{\mathrm{e}^{\mathcal{G}}[\mathbf{p}^2-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)(\sigma_1\cdot\mathbf{p})$$

$$-\frac{i}{2}\nabla\mathcal{G}\cdot(\mathbf{p}+i(\sigma_1\times\mathbf{p})-(\sigma_1\cdot\sigma_2)\mathbf{p}+\sigma_1(\sigma_2\cdot\mathbf{p})-i(\sigma_2\times\mathbf{p}))]$$

$$+\frac{E_1}{i}\sigma_1\cdot\nabla[\mathrm{e}^{\mathcal{G}}\frac{M_2}{\mathcal{D}}[\sigma_1\cdot\mathbf{p}-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]]$$

$$+\frac{M_1}{i}\sigma_1\cdot\nabla[\mathrm{e}^{\mathcal{G}}\frac{E_2}{\mathcal{D}}[\sigma_1\cdot\mathbf{p}-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]]$$

$$-\frac{iE_1}{2}[\sigma_2\cdot\nabla(C+J+L)+\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]\mathrm{e}^{\mathcal{G}}\frac{M_2}{\mathcal{D}}[\sigma_1\cdot\mathbf{p}$$

$$-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]$$

$$-\frac{iM_1}{2}[\sigma_2\cdot\nabla(C-J-L)+\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]\mathrm{e}^{\mathcal{G}}\frac{E_2}{\mathcal{D}}[\sigma_1\cdot\mathbf{p}$$

$$-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]\}$$

$$+\mathrm{e}^{\mathcal{G}}\{\frac{M_1}{i}\sigma_1\cdot\nabla[\mathrm{e}^{\mathcal{G}}\frac{E_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_2+i\sigma_2\times\sigma_1)]]$$

$$-\frac{E_1}{i}\sigma_1\cdot\nabla[\mathrm{e}^{\mathcal{G}}\frac{M_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_2+i\sigma_2\times\sigma_1)]]$$

$$+i\sigma_2\cdot\nabla(J+L)[\mathrm{e}^{\mathcal{G}}\frac{M_1E_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_2+i\sigma_2\times\sigma_1)]]\}$$

applying $\nabla\psi\phi=\psi\nabla\phi+\phi\nabla\psi$ to the second and third terms of above equation, and using $\mathcal{D}=E_1M_2+E_2M_1$ to combine them, we can get three terms from this combination. Also using $\mathcal{D}=E_1M_2+E_2M_1$ to combine the terms with $\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)$ of 4th and 5th terms of above equation. Keep all the other term unchanged, we obtain

(a) + (b) + (c) + (d) term=

$$\mathrm{e}^{\mathcal{G}}\{\mathrm{e}^{\mathcal{G}}[\mathbf{p}^2-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)(\sigma_1\cdot\mathbf{p})$$

$$-\frac{i}{2}\nabla\mathcal{G}\cdot(\mathbf{p}-(\sigma_1\cdot\sigma_2)\mathbf{p}+\sigma_1(\sigma_2\cdot\mathbf{p})-i(\sigma_2-\sigma_1)\times\mathbf{p})]$$

$$+\frac{e^{\mathcal{G}}}{i}(\sigma_1\cdot\nabla)[\sigma_1\cdot\mathbf{p}-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]$$

$$+\frac{E_1}{i}\sigma_1\cdot\nabla(e^{\mathcal{G}}\frac{M_2}{\mathcal{D}})[\sigma_1\cdot\mathbf{p}-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]]$$

$$+\frac{M_1}{i}\sigma_1\cdot\nabla(e^{\mathcal{G}}\frac{E_2}{\mathcal{D}})[\sigma_1\cdot\mathbf{p}-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]]$$

$$-\frac{i}{2}e^{\mathcal{G}}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)[\sigma_1\cdot\mathbf{p}-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]$$

$$-\frac{iE_1}{2}e^{\mathcal{G}}\frac{M_2}{\mathcal{D}}\sigma_2\cdot\nabla(C+J+L)[\sigma_1\cdot\mathbf{p}-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]$$

$$-\frac{iM_1}{2}e^{\mathcal{G}}\frac{E_2}{\mathcal{D}}\sigma_2\cdot\nabla(C-J-L)[\sigma_1\cdot\mathbf{p}-\frac{i}{2}\sigma_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_1+i\sigma_1\times\sigma_2)]\}$$

$$+e^{\mathcal{G}}\{\frac{M_1}{i}\sigma_1\cdot\nabla[e^{\mathcal{G}}\frac{E_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_2+i\sigma_2\times\sigma_1)]]$$

$$-\frac{E_1}{i}\sigma_1\cdot\nabla[e^{\mathcal{G}}\frac{M_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_2+i\sigma_2\times\sigma_1)]]$$

$$+i\sigma_2\cdot\nabla(J+L)[e^{\mathcal{G}}\frac{M_1E_1}{\mathcal{D}}[\sigma_2\cdot\mathbf{p}-\frac{i}{2}\sigma_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\sigma_2+i\sigma_2\times\sigma_1)]]\}$$

Simplify above expression using $(\sigma\cdot\mathbf{A})(\sigma\cdot\mathbf{B})=\mathbf{A}\cdot\mathbf{B}+\mathbf{i}\sigma\cdot(\mathbf{A}\times\mathbf{B})$, we obtain

(a) + (b) + (c) + (d) term=

$$\mathrm{e}^{\mathcal{G}}\{\mathrm{e}^{\mathcal{G}}[\mathbf{p}^2 - \frac{i}{2}\sigma_2 \cdot \nabla(-C + J - L)(\sigma_1 \cdot \mathbf{p})$$

$$-\frac{i}{2}\nabla\mathcal{G} \cdot (\mathbf{p} - (\sigma_1 \cdot \sigma_2)\mathbf{p} + \sigma_1(\sigma_2 \cdot \mathbf{p}) - i(\sigma_2 - \sigma_1) \times \mathbf{p})]$$

$$+\frac{\mathrm{e}^{\mathcal{G}}}{i}[\nabla \cdot \mathbf{p} + \mathbf{i}\sigma_1 \cdot (\nabla \times \mathbf{p}) - \frac{i}{2}(\sigma_1 \cdot \nabla)\sigma_2 \cdot \nabla(-C + J - L) - \frac{i}{2}\nabla^2\mathcal{G}$$

$$+\frac{1}{2}\nabla \cdot (i\nabla\mathcal{G}(\sigma_1 \cdot \sigma_2) - i\sigma_2(\sigma_1 \cdot \nabla\mathcal{G}) - \nabla\mathcal{G}\times\sigma_2)]$$

$$+[\frac{E_1}{i}(\mathrm{e}^{\mathcal{G}}\frac{M_2}{\mathcal{D}})' + \frac{M_1}{i}(\mathrm{e}^{\mathcal{G}}\frac{E_2}{\mathcal{D}})'][\hat{\mathbf{r}} \cdot \mathbf{p} + i\sigma_1 \cdot (\hat{\mathbf{r}} \times \mathbf{p}) - \frac{i}{2}(\sigma_1 \cdot \hat{\mathbf{r}})\sigma_2 \cdot \nabla(-C + J - L)$$

$$-\frac{i}{2}(\hat{\mathbf{r}}\cdot\nabla\mathcal{G} + i\sigma_1 \cdot (\hat{\mathbf{r}}\times\nabla\mathcal{G})) + \frac{1}{2}\hat{\mathbf{r}} \cdot (i\nabla\mathcal{G}(\sigma_1 \cdot \sigma_2) - i\sigma_2(\sigma_1 \cdot \nabla\mathcal{G}) - \nabla\mathcal{G}\times\sigma_2)]$$

$$-\frac{i}{2}\mathrm{e}^{\mathcal{G}}[\nabla\mathcal{G} \cdot \mathbf{p} + i\sigma_1 \cdot (\nabla\mathcal{G}\times\mathbf{p}) + i(-i\nabla\mathcal{G}(\sigma_1 \cdot \sigma_2) + i\sigma_2(\sigma_1 \cdot \nabla\mathcal{G}) - \nabla\mathcal{G}\times\sigma_2) \cdot \mathbf{p}$$

$$-\frac{i}{2}(\sigma_1 \cdot \nabla\mathcal{G})(\sigma_2 \cdot \nabla(-C + J - L))$$

$$-\frac{1}{2}(-i\nabla\mathcal{G}(\sigma_1 \cdot \sigma_2) + i\sigma_1(\sigma_2 \cdot \nabla\mathcal{G}) - \nabla\mathcal{G}\times\sigma_1) \cdot \nabla(-C + J - L)$$

$$-\frac{i}{2}((\nabla\mathcal{G})^2-(\nabla\mathcal{G}\cdot(\sigma_1\times\sigma_2))^2+i(\nabla\mathcal{G}\cdot(i\nabla\mathcal{G}(\sigma_1\cdot\sigma_2)-i\sigma_2(\sigma_1\cdot\nabla\mathcal{G})-\nabla\mathcal{G}\times\sigma_2)))$$

$$+i(-i\nabla\mathcal{G}(\sigma_1\cdot\sigma_2)+i\sigma_2(\sigma_1\cdot\nabla\mathcal{G})-\nabla\mathcal{G}\times\sigma_2)\cdot\nabla\mathcal{G}]$$

$$-\frac{iE_1}{2}\mathrm{e}^{\mathcal{G}}\frac{M_2}{\mathcal{D}}[\sigma_2\cdot\nabla(C+J+L)(\sigma_1\cdot\mathbf{p})$$

$$-\frac{i}{2}\nabla(C+J+L)\cdot\nabla(-C+J-L)-\frac{i}{2}\sigma_2\cdot\nabla(C+J+L)(\sigma_1\cdot\nabla\mathcal{G})$$

$$-\frac{1}{2}\nabla(C+J+L)\cdot(i\nabla\mathcal{G}(\sigma_1\cdot\sigma_2)-i\sigma_1(\sigma_2\cdot\nabla\mathcal{G})-\nabla\mathcal{G}\times\sigma_1)]$$

$$-\frac{iM_1}{2}\mathrm{e}^{\mathcal{G}}\frac{E_2}{\mathcal{D}}[\sigma_2\cdot\nabla(C-J-L)(\sigma_1\cdot\mathbf{p})$$

$$-\frac{i}{2}\nabla(C-J-L)\cdot\nabla(-C+J-L)-\frac{i}{2}\sigma_2\cdot\nabla(C-J-L)(\sigma_1\cdot\nabla\mathcal{G})$$

$$-\frac{1}{2}\nabla(C-J-L)\cdot(i\nabla\mathcal{G}(\sigma_1\cdot\sigma_2)-i\sigma_1(\sigma_2\cdot\nabla\mathcal{G})-\nabla\mathcal{G}\times\sigma_1)]\}$$

$$+\mathrm{e}^{\mathcal{G}}\{[\frac{M_1}{i}(\mathrm{e}^{\mathcal{G}}\frac{E_1}{\mathcal{D}})'-\frac{E_1}{i}(\mathrm{e}^{\mathcal{G}}\frac{M_1}{\mathcal{D}})'][(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\mathbf{p})$$

$$-\frac{i}{2}(\hat{\mathbf{r}}\cdot\nabla(-C+J-L)+i\sigma_1\cdot(\hat{\mathbf{r}}\times\nabla(-C+J-L))$$

$$-\frac{i}{2}(\sigma_1 \cdot \hat{\mathbf{r}})(\sigma_2 \cdot \nabla\mathcal{G}) - \frac{1}{2}\hat{\mathbf{r}} \cdot (i\nabla\mathcal{G}(\sigma_1 \cdot \sigma_2) - i\sigma_2(\sigma_1 \cdot \nabla\mathcal{G}) - \nabla\mathcal{G}\times\sigma_2)]$$

$$+i\mathrm{e}^{\mathcal{G}}\frac{M_1 E_1}{\mathcal{D}}[\nabla(J+L) \cdot \mathbf{p} - i\sigma_2 \cdot (\nabla(J+L) \times \mathbf{p})$$

$$-\frac{i}{2}(\sigma_2 \cdot \nabla(J+L))(\sigma_1 \cdot \nabla(-C+J-L))$$

$$-\frac{i}{2}(\nabla(J+L) \cdot \nabla\mathcal{G} + i\sigma_2 \cdot (\nabla(J+L) \times \nabla\mathcal{G}))$$

$$+\frac{1}{2}\nabla(J+L) \cdot (i\nabla\mathcal{G}(\sigma_1 \cdot \sigma_2) - i\sigma_1(\sigma_2 \cdot \nabla\mathcal{G}) - \nabla\mathcal{G}\times\sigma_1)]\}$$

Simplifying above expression by letting

$$\nabla(C+J+L) = (C+J+L)'\hat{\mathbf{r}},$$

$$\nabla(-C+J-L) = (-C+J-L)'\hat{\mathbf{r}},$$

$$\nabla\mathcal{G} = \mathcal{G}'\hat{\mathbf{r}}, etc.$$

and using

$$\nabla^2 f(r) = f(r)'' + \frac{2}{r}f(r)'$$

$$(\nabla\mathcal{G}\cdot(\sigma_1\times\sigma_2))^2 = 2(\nabla\mathcal{G})^2 - 2\mathcal{G}'^2(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}})$$

some terms may become zero because the vector dot and cross product properties. The above expression become

(a) + (b) + (c) + (d) term=

$$\mathrm{e}^{\mathcal{G}}\{\mathrm{e}^{\mathcal{G}}[\mathbf{p}^2 - \frac{i}{2}(-C+J-L)'(\sigma_2\cdot\hat{\mathbf{r}})(\sigma_1\cdot\mathbf{p})$$

$$-\frac{i}{2}\mathcal{G}'(\hat{\mathbf{r}}\cdot\mathbf{p} - (\sigma_1\cdot\sigma_2)\hat{\mathbf{r}}\cdot\mathbf{p} + (\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\mathbf{p}) - i\hat{\mathbf{r}}\cdot((\sigma_2-\sigma_1)\times\mathbf{p}))]$$

$$+\frac{\mathrm{e}^{\mathcal{G}}}{i}[-\frac{i}{2}\nabla^2(-C+J-L)(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}}) + \frac{3i}{2}\frac{(-C+J-L)'}{r}(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}})$$

$$-\frac{i}{2}(-C+J-L)'(\sigma_1\cdot\sigma_2) - \frac{i}{2}\nabla^2\mathcal{G}$$

$$+\frac{1}{2}(i\nabla^2\mathcal{G}(\sigma_1\cdot\sigma_2) - i\nabla^2\mathcal{G}(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}}) + \frac{3i}{2}\frac{\mathcal{G}'}{r}(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}}) - \frac{i}{2}\mathcal{G}'(\sigma_1\cdot\sigma_2))]$$

$$+\frac{\mathrm{e}^{\mathcal{G}}}{i}[\mathcal{G}' - \frac{E_2M_2}{\mathcal{D}}(J+L)'][\hat{\mathbf{r}}\cdot\mathbf{p} + i\sigma_1\cdot(\hat{\mathbf{r}}\times\mathbf{p}) - \frac{i}{2}(-C+J-L)'(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}})$$

$$-\frac{i}{2}\mathcal{G}' + \frac{i}{2}\mathcal{G}'(\sigma_1\cdot\sigma_2) - \frac{i}{2}\mathcal{G}'(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}})]$$

$$-\frac{i}{2}\mathrm{e}^{\mathcal{G}}[\mathcal{G}'\hat{\mathbf{r}}\cdot\mathbf{p}+i\mathcal{G}'\sigma_1\cdot(\hat{\mathbf{r}}\times\mathbf{p})+\mathcal{G}'(\sigma_1\cdot\sigma_2)\hat{\mathbf{r}}\cdot\mathbf{p}-\mathcal{G}'(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\mathbf{p})-i\mathcal{G}'(\hat{\mathbf{r}}\times\sigma_2)\cdot\mathbf{p}$$

$$-\frac{i}{2}\mathcal{G}'(-C+J-L)'(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}})+\frac{i}{2}\mathcal{G}'(-C+J-L)'(\sigma_1\cdot\sigma_2)$$

$$-\frac{i}{2}\mathcal{G}'(-C+J-L)'(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}})-\frac{i}{2}((\nabla\mathcal{G})^2-2(\nabla\mathcal{G})^2+2\mathcal{G}'^2(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}})$$

$$-\mathcal{G}'^2(\sigma_1\cdot\sigma_2)+\mathcal{G}'^2(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}})+\mathcal{G}'^2(\sigma_1\cdot\sigma_2)-\mathcal{G}'^2(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}}))]$$

$$-\frac{iE_1}{2}\mathrm{e}^{\mathcal{G}}\frac{M_2}{\mathcal{D}}[(C+J+L)'(\sigma_2\cdot\hat{\mathbf{r}})(\sigma_1\cdot\mathbf{p})-\frac{i}{2}(C+J+L)'(-C+J-L)'$$

$$-\frac{i}{2}\mathcal{G}'(C+J+L)'(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}})$$

$$-\frac{i}{2}(C+J+L)'\mathcal{G}'(\sigma_1\cdot\sigma_2)+\frac{i}{2}\mathcal{G}'(C+J+L)'(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}})]$$

$$-\frac{iM_1}{2}\mathrm{e}^{\mathcal{G}}\frac{E_2}{\mathcal{D}}[(C-J-L)'(\sigma_2\cdot\hat{\mathbf{r}})(\sigma_1\cdot\mathbf{p})-\frac{i}{2}(C-J-L)'(-C+J-L)'$$

$$-\frac{i}{2}(C-J-L)'\mathcal{G}'(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}})$$

$$-\frac{i}{2}(C-J-L)'\mathcal{G}'(\sigma_1\cdot\sigma_2)+\frac{i}{2}(C-J-L)'\mathcal{G}'(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}})]\}$$

$$+\mathrm{e}^{\mathcal{G}}\{\frac{\mathrm{e}^{\mathcal{G}}}{i}[J' - \frac{E_1 M_2}{\mathcal{D}}(J+L)'][(\sigma_1 \cdot \hat{\mathbf{r}})(\sigma_2 \cdot \mathbf{p}) - \frac{i}{2}(-C+J-L)'$$

$$-\frac{i}{2}\mathcal{G}'(\sigma_1 \cdot \hat{\mathbf{r}})(\sigma_2 \cdot \hat{\mathbf{r}}) - \frac{i}{2}\mathcal{G}'(\sigma_1 \cdot \sigma_2) + \frac{i}{2}\mathcal{G}'(\sigma_1 \cdot \hat{\mathbf{r}})(\sigma_2 \cdot \hat{\mathbf{r}})]$$

$$+i\mathrm{e}^{\mathcal{G}}\frac{E_1 M_1}{\mathcal{D}}[(J+L)'\hat{\mathbf{r}} \cdot \mathbf{p} + \mathbf{i}(J+L)'\sigma_2 \cdot (\hat{\mathbf{r}} \times \mathbf{p})$$

$$-\frac{i}{2}(J+L)'(-C+J-L)'(\sigma_1 \cdot \hat{\mathbf{r}})(\sigma_2 \cdot \hat{\mathbf{r}})$$

$$-\frac{i}{2}\mathcal{G}'(J+L)' + \frac{i}{2}\mathcal{G}'(J+L)'(\sigma_1 \cdot \sigma_2) - \frac{i}{2}\mathcal{G}'(J+L)'(\sigma_1 \cdot \hat{\mathbf{r}})(\sigma_2 \cdot \hat{\mathbf{r}})]\}$$

Some terms in above expression can be combined, some cancel each other. Finally, we can group above equations by $\mathbf{p}^2$term , Darwin term $\hat{\mathbf{r}} \cdot \mathbf{p}$, spin-orbit angular momentum term $L \cdot (\sigma_1 + \sigma_2)$, spin-orbit angular momentum difference term $L \cdot (\sigma_1 - \sigma_2)$, spin-spin term $(\sigma_1 \cdot \sigma_2)$, tensor term $(\sigma_1 \cdot \hat{\mathbf{r}})(\sigma_2 \cdot \hat{\mathbf{r}})$, additional spin dependent terms $L \cdot (\sigma_1 \times \sigma_2)$ and $(\sigma_1 \cdot \hat{\mathbf{r}})(\sigma_2 \cdot \mathbf{p}) + (\sigma_2 \cdot \hat{\mathbf{r}})(\sigma_1 \cdot \mathbf{p})$ and spin independent terms. The final result for above expression is

(a) + (b) + (c) + (d) term=

$$\mathrm{e}^{2\mathcal{G}}\{\mathbf{p}^2 - i[2\mathcal{G}' - \frac{E_2 M_2 + M_1 E_1}{\mathcal{D}}(J+L)']\hat{\mathbf{r}} \cdot \mathbf{p} - \frac{1}{2}\nabla^2\mathcal{G} - \frac{1}{4}\mathcal{G}'^2$$

$$-\frac{1}{4}((J-L)'^2 - C'^2) + \frac{1}{2}\frac{E_2M_2 + M_1E_1}{\mathcal{D}}\mathcal{G}'(J+L)'$$

$$+\frac{L\cdot(\sigma_1+\sigma_2)}{r}[\mathcal{G}' - \frac{1}{2}\frac{E_2M_2+M_1E_1}{\mathcal{D}}(J+L)'] - \frac{L\cdot(\sigma_1-\sigma_2)}{r}\frac{1}{2}\frac{E_2M_2-M_1E_1}{\mathcal{D}}(J+L)'$$

$$+(\sigma_1\cdot\sigma_2)[\frac{1}{2}\nabla^2\mathcal{G}+\frac{1}{2}\mathcal{G}'^2 - \frac{1}{2}\frac{E_2M_2+M_1E_1}{\mathcal{D}}\mathcal{G}'(J+L)' - \frac{1}{2}\mathcal{G}'C' - \frac{1}{2}\frac{\mathcal{G}'}{r} - \frac{1}{2}\frac{(-C+J-L)'}{r}]$$

$$+(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}})[-\frac{1}{2}\nabla^2(-C+J-L) - \frac{1}{2}\nabla^2\mathcal{G} - \mathcal{G}'(-C+J-L)' - \mathcal{G}'^2 + \frac{3}{2r}\mathcal{G}'$$

$$+\frac{3}{2r}(-C+J-L)' + \frac{1}{2}\frac{E_2M_2+M_1E_1}{\mathcal{D}}(J+L)'(\mathcal{G}-C+J-L)']$$

$$+\frac{L\cdot(\sigma_1\times\sigma_2)}{r}\frac{i}{2}\frac{M_2E_1-M_1E_2}{\mathcal{D}}(J+L)' - ((\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\mathbf{p}) + (\sigma_2\cdot\hat{\mathbf{r}})(\sigma_1\cdot\mathbf{p}))\frac{i(J-L)'}{2}\}$$

So our Eq.(B.1) becomes

$$\{\mathbf{p}^2 - i[2\mathcal{G}' - \frac{E_2M_2+M_1E_1}{\mathcal{D}}(J+L)']\hat{\mathbf{r}}\cdot\mathbf{p} - \frac{1}{2}\nabla^2\mathcal{G} - \frac{1}{4}\mathcal{G}'^2$$

$$-\frac{1}{4}((J-L)'^2 - C'^2) + \frac{1}{2}\frac{E_2M_2 + M_1E_1}{\mathcal{D}}\mathcal{G}'(J+L)'$$

$$+\frac{L\cdot(\sigma_1+\sigma_2)}{r}[\mathcal{G}' - \frac{1}{2}\frac{E_2M_2+M_1E_1}{\mathcal{D}}(J+L)'] - \frac{L\cdot(\sigma_1-\sigma_2)}{r}\frac{1}{2}\frac{E_2M_2-M_1E_1}{\mathcal{D}}(J+L)'$$

$$+(\sigma_1\cdot\sigma_2)[\frac{1}{2}\nabla^2\mathcal{G}+\frac{1}{2}\mathcal{G}'^2 - \frac{1}{2}\frac{E_2M_2+M_1E_1}{\mathcal{D}}\mathcal{G}'(J+L)' - \frac{1}{2}\mathcal{G}'C' - \frac{1}{2}\frac{\mathcal{G}'}{r} - \frac{1}{2}\frac{(-C+J-L)'}{r}]$$

$$+(\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\hat{\mathbf{r}})[-\frac{1}{2}\nabla^2(-C+J-L) - \frac{1}{2}\nabla^2\mathcal{G} - \mathcal{G}'(-C+J-L)' - \mathcal{G}'^2 + \frac{3}{2r}\mathcal{G}'$$

$$+\frac{3}{2r}(-C+J-L)' + \frac{1}{2}\frac{E_2M_2+M_1E_1}{\mathcal{D}}(J+L)'(\mathcal{G}-C+J-L)']$$

$$+\frac{L\cdot(\sigma_1\times\sigma_2)}{r}\frac{i}{2}\frac{M_2E_1-M_1E_2}{\mathcal{D}}(J+L)' - ((\sigma_1\cdot\hat{\mathbf{r}})(\sigma_2\cdot\mathbf{p}) + (\sigma_2\cdot\hat{\mathbf{r}})(\sigma_1\cdot\mathbf{p}))\frac{i(J-L)'}{2}\}\phi_+$$

$$= \mathcal{B}^2 \mathrm{e}^{-2\mathcal{G}}\phi_+. \qquad \text{(B.11)}$$

This is our Schrödinger-like equation which we will use to fit phase shift experimental data. We can obtain the wave equations for different states from this equation.

B.2 The Radial Eigenvalue Equations

For singlet states $^1S_0$, $^1P_1$, $^1D_2$ and triplet states $^3P_0$, $^3P_1$, $^3S_1$, $^3D_1$, we can get their radial eigenvalue equations from Eq.(B.11) for equal mass case as following

$s = 0, j = l$

$$\{-\frac{d^2}{dr^2} + \frac{j(j+1)}{r^2} - (2\mathcal{G} - \ln(\mathcal{D}) - J + L)'(\frac{d}{dr} - \frac{1}{r})$$

$$+\frac{1}{2}\nabla^2(-C + J - L - 3\mathcal{G}) - \frac{1}{4}(C + J - L - \mathcal{G} + 2\ln(\mathcal{D}))'(-C + J - L - 3\mathcal{G})'\}u_{j0j}$$

$$= \mathcal{B}^2 e^{-2\mathcal{G}} u_{j0j},$$

$s = 1, j = l$

$$\{-\frac{d^2}{dr^2} + \frac{j(j+1)}{r^2} - (2\mathcal{G} + J - L - \ln(\mathcal{D}))'\frac{d}{dr} + \frac{(-C + J - L + \mathcal{G})'}{r}$$

$$-\frac{1}{2}\nabla^2(-C + J - L + \mathcal{G}) + \frac{1}{4}(2\ln(\mathcal{D}) - (C + J - L + 3\mathcal{G}))'(-C + J - L + \mathcal{G})'\}u_{j1j}$$

$$= \mathcal{B}^2 e^{-2\mathcal{G}} u_{j1j}.$$

The above two equations are uncoupled. The below are the coupled equations

$s = 1, j = l + 1$

$$\{(-\frac{d^2}{dr^2} + \frac{j(j-1)}{r^2}) + [\ln(\mathcal{D}) - 2\mathcal{G} - \frac{1}{2j+1}(J-L)]'\frac{d}{dr}$$

$$[-j\ln(\mathcal{D}) + \frac{1}{2j+1}((4j^2+j+1)\mathcal{G} + J - L + (j-1)C)]'\frac{1}{r} - \frac{1}{2}\mathcal{G}'C' + \frac{1}{4}(C'^2 - (J-L)'^2)$$

$$+\frac{1}{2j+1}((-\frac{1}{2}\nabla^2(J-L+\mathcal{G}-C)+\mathcal{G}'(\frac{2j-3}{4}\mathcal{G}-J+L+C)'+\frac{1}{2}\ln'(\mathcal{D})(\mathcal{G}+J-L-C)')\}u_-$$

$$+\frac{\sqrt{j(j+1)}}{2j+1}\{-2[J-L]'\frac{d}{dr} + [(J-L)(1-2j) + 3\mathcal{G} - 3C]'\frac{1}{r}$$

$$-\nabla^2(J-L+\mathcal{G}-C) + (J-L+\mathcal{G}-C)'(\ln(\mathcal{D}) - 2\mathcal{G})'\}u_+ = \mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}u_-,$$

$s = 1, j = l - 1$

$$\{(-\frac{d^2}{dr^2} + \frac{(j+1)(j+2)}{r^2}) + [\ln(\mathcal{D}) - 2\mathcal{G} + \frac{1}{2j+1}(J-L)]'\frac{d}{dr}$$

$$[(j+1)\ln(\mathcal{D}) - \frac{1}{2j+1}((4j^2+7j+4)\mathcal{G} + J - L - (j+2)C)]'\frac{1}{r} - \frac{1}{2}\mathcal{G}'C' + \frac{1}{4}(C'^2 - (J-L)'^2)$$

$$+\frac{1}{2j+1}((\frac{1}{2}\nabla^2(J-L+\mathcal{G}-C)+\mathcal{G}'(\frac{2j+5}{4}\mathcal{G}+J-L-C)'-\frac{1}{2}\ln'(\mathcal{D})(\mathcal{G}+J-L-C)')\}u_+$$

$$+\frac{\sqrt{j(j+1)}}{2j+1}\{-2[J-L]'\frac{d}{dr}+[(J-L)(2j+3)+3\mathcal{G}-3C]'\frac{1}{r}$$

$$-\nabla^2(J-L+\mathcal{G}-C)+(J-L+\mathcal{G}-C)'(\ln(\mathcal{D})-2\mathcal{G})'\}u_- = \mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}u_+.$$

The above radial eigenvalue equations agree with Long's results[31], but by using the different methods for derivation. We do not give the explicit detail of this derivation since the method we used here are similar to one we use in Appendix C which we use for our final equations . All of above equations have a first derivative term. This is a difference with the standard Schrödinger-like equation. Before we can apply the techniques which have been already developed for the Schrödinger-like system in nonrelativistic quantum mechanics, we must get rid of these first derivative terms .

Appendix C

Removal Of The First Derivative Terms

C.1 Removal Of The First Derivative Terms

The general form of the eigenvalue equation given in Eq.(3.175) is:

$$[\mathbf{p}^2 - ig'\hat{\mathbf{r}} \cdot \mathbf{p} + \frac{g'}{2r}\vec{L} \cdot (\sigma_1 + \sigma_2) - ih'(\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p})$$

$$+k\sigma_1 \cdot \sigma_2 + j\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}} + m]\Psi = \mathcal{B}^2 e^{-2\mathcal{G}}\Psi. \tag{C.1}$$

The m term is the spin independent term. We let

$$\Psi = \exp(F + K\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\psi \equiv (A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\psi. \tag{C.2}$$

Next we want to move all the $\mathbf{p}$ terms in above equations to the right side of $(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})$. We begin with

$$\mathbf{p}\Psi = \mathbf{p}(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\psi$$

and by using the commutation relation of $\mathbf{p}$, we can move $\mathbf{p}$ to the right side, so

$$\mathbf{p}\Psi = (A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\mathbf{p}\psi - i\partial(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\psi$$

$$= (A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\mathbf{p}\psi - i(A' + B'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}\psi - iB\partial[(\sigma_1 \cdot \hat{\mathbf{r}})(\sigma_2 \cdot \hat{\mathbf{r}})]\psi$$

Finally,

$$\begin{aligned}\mathbf{p}\Psi &= (A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\mathbf{p}\psi - i(A' + B'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}\psi \\ &\quad -i\frac{B}{r}[(\sigma_1 - \sigma_1 \cdot \hat{\mathbf{r}}\hat{\mathbf{r}})\sigma_2 \cdot \hat{\mathbf{r}} + (\sigma_2 - \sigma_2 \cdot \hat{\mathbf{r}}\hat{\mathbf{r}})\sigma_1 \cdot \hat{\mathbf{r}}]\psi, \end{aligned} \tag{C.3}$$

$$\frac{g'}{2r}\vec{L} \cdot (\sigma_1 + \sigma_2)\Psi = \frac{g'}{2r}\vec{L} \cdot (\sigma_1 + \sigma_2)(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\psi$$

$$= (A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\frac{g'}{2r}\vec{L} \cdot (\sigma_1 + \sigma_2)\psi + \frac{g'}{2r}[\vec{L} \cdot (\sigma_1 + \sigma_2), (A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})]\psi$$

$$= (A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\frac{g'}{2r}\vec{L} \cdot (\sigma_1 + \sigma_2)\psi$$

$$+\frac{g'}{2r}B[2\sigma_1 \cdot \sigma_2 - 4ir\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p} + 2ir(\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p}) - 6\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}]\psi, \tag{C.4}$$

by making use of Eq.(C.3), we find that

$$-ig'\hat{\mathbf{r}}\cdot\mathbf{p}\Psi = (A + B\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})(-ig'\hat{\mathbf{r}}\cdot\mathbf{p})\psi + C\psi \qquad \text{(C.5)}$$

where

$$C = -g'(A' + B'\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}),$$

and

$$\begin{aligned}-ih'(\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\mathbf{p} + \sigma_2\cdot\hat{\mathbf{r}}\sigma_1\cdot\mathbf{p})\Psi &= (A + B\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})(-ih'[\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\mathbf{p} + \sigma_2\cdot\hat{\mathbf{r}}\sigma_1\cdot\mathbf{p}])\psi \\ &\quad +D\psi\end{aligned}$$

where

$$\begin{aligned}D &= -ih'[\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\mathbf{p} + \sigma_2\cdot\hat{\mathbf{r}}\sigma_1\cdot\mathbf{p}, A + B\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}] \\ &= -2h'(A'\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}} + \sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}B'\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}) \\ &\quad +h'B\{(\sigma_1\cdot\hat{\mathbf{r}})[-\frac{1}{r}\sigma_2\cdot((\boldsymbol{\sigma_1} - \boldsymbol{\sigma_1}\cdot\hat{\mathbf{r}}\hat{\mathbf{r}})\boldsymbol{\sigma_2}\cdot\hat{\mathbf{r}} + (\boldsymbol{\sigma_2} - \boldsymbol{\sigma_2}\cdot\hat{\mathbf{r}}\hat{\mathbf{r}})\boldsymbol{\sigma_1}\cdot\hat{\mathbf{r}}) - i[\boldsymbol{\sigma_2}, \sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}]\mathbf{p}] \\ &\quad +(\sigma_2\cdot\hat{\mathbf{r}})[-\frac{1}{r}\sigma_1\cdot((\boldsymbol{\sigma_1} - \boldsymbol{\sigma_1}\cdot\hat{\mathbf{r}}\hat{\mathbf{r}})\boldsymbol{\sigma_2}\cdot\hat{\mathbf{r}} + (\boldsymbol{\sigma_2} - \boldsymbol{\sigma_2}\cdot\hat{\mathbf{r}}\hat{\mathbf{r}})\boldsymbol{\sigma_1}\cdot\hat{\mathbf{r}}) - i[\boldsymbol{\sigma_1}, \sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}]\mathbf{p}]\}\end{aligned}$$

$$
\begin{aligned}
&= -2h'(A'\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}+B') \\
&\quad +h'B\{(\sigma_1\cdot\hat{\mathbf{r}})[-\frac{1}{r}(\sigma_{\mathbf{2}}\cdot\sigma_{\mathbf{1}}\sigma_{\mathbf{2}}\cdot\hat{\mathbf{r}}-\sigma_1\cdot\hat{\mathbf{r}}+2\sigma_1\cdot\hat{\mathbf{r}})-2\frac{\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot L}{r}] \\
&\quad +(\sigma_2\cdot\hat{\mathbf{r}})[-\frac{1}{r}(\sigma_{\mathbf{2}}\cdot\sigma_{\mathbf{1}}\sigma_{\mathbf{1}}\cdot\hat{\mathbf{r}}-\sigma_2\cdot\hat{\mathbf{r}}+2\sigma_2\cdot\hat{\mathbf{r}})-2\frac{\sigma_2\cdot\hat{\mathbf{r}}\sigma_1\cdot L}{r}]\} \\
&= -2h'(A'\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}+B') \\
&\quad +h'B\{-\frac{1}{r}[\sigma_1\cdot\hat{\mathbf{r}}\sigma_{\mathbf{2}}\cdot\sigma_{\mathbf{1}}\sigma_{\mathbf{2}}\cdot\hat{\mathbf{r}}+\sigma_{\mathbf{2}}\cdot\hat{\mathbf{r}}\sigma_{\mathbf{2}}\cdot\sigma_{\mathbf{1}}\sigma_{\mathbf{1}}\cdot\hat{\mathbf{r}}+2]-2\frac{(\sigma_1+\sigma_2)\cdot L}{r}\} \\
&= -2h'(A'\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}+B')+2h'B\{-\frac{1}{r}[2-\sigma_1\cdot\hat{\mathbf{r}}\sigma_{\mathbf{2}}\cdot\hat{\mathbf{r}}+\sigma_{\mathbf{2}}\cdot\sigma_{\mathbf{1}}]-\frac{(\sigma_1+\sigma_2)\cdot L}{r}\} \\
&= -2h'(\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}A'+B')-2h'\frac{B}{r}[\vec{L}\cdot(\sigma_1+\sigma_2)+2-\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}+\sigma_1\cdot\sigma_2],
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{p}^2\Psi &= \mathbf{p}^2(A+B\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})\psi \\
&= (A+B\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})\mathbf{p}^2\psi+[\mathbf{p}^2,A+B\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}]\psi
\end{aligned}
$$

$$
\begin{aligned}
&= (A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\mathbf{p}^2\psi + \mathbf{p}[\mathbf{p}, A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}]\psi + [\mathbf{p}, A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}]\mathbf{p}\psi \\
&= (A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\mathbf{p}^2\psi + [\mathbf{p},[\mathbf{p}, A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}]]\psi \\
&\quad +2[\mathbf{p}, A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}]\mathbf{p}\psi \\
&= (A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\mathbf{p}^2\psi - 2i(A' + B'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} \cdot \mathbf{p}\psi \\
&\quad +i\frac{2B}{r}[2\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p} - (\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p}]\psi + E\psi \qquad \text{(C.6)}
\end{aligned}
$$

where

$$
\begin{aligned}
E &= [\mathbf{p},[\mathbf{p}, A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}]] \\
&= [\mathbf{p}, -i(A' + B'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - i\frac{B}{r}[(\sigma_1 - \sigma_1 \cdot \hat{\mathbf{r}}\hat{\mathbf{r}})\sigma_2 \cdot \hat{\mathbf{r}} + (\sigma_2 - \sigma_2 \cdot \hat{\mathbf{r}}\hat{\mathbf{r}})\sigma_1 \cdot \hat{\mathbf{r}}]] \\
&= -(A'' + B''\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}) - \frac{2}{r}(A' + B'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}) \\
&\quad -\frac{B}{r}(\frac{2(\sigma_1 - \sigma_1 \cdot \hat{\mathbf{r}}\hat{\mathbf{r}})(\sigma_2 - \sigma_2 \cdot \hat{\mathbf{r}}\hat{\mathbf{r}})}{r} - \frac{4\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}}{r})
\end{aligned}
$$

$$= \quad -(A'' + B''\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}) - \frac{2}{r}(A' + B'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}) - 2\frac{B}{r^2}(\sigma_1 \cdot \sigma_2 - 3\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}).$$

Note C and D and E do not involve $\mathbf{p}$. The Eq.(C.1) becomes

$$(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})[\mathbf{p}^2 - ig'\hat{\mathbf{r}} \cdot \mathbf{p} + \frac{g'}{2r}\vec{L} \cdot (\sigma_1 + \sigma_2) - ih'(\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p})]\psi$$

$$+(\frac{g'}{2r}B[2\sigma_1 \cdot \sigma_2 - 4ir\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p} + 2ir(\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p}) - 6\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}]$$

$$-2i(A' + B'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} \cdot \mathbf{p}$$

$$+i\frac{2B}{r}[2\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p} - (\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p})]$$

$$+(k\sigma_1 \cdot \sigma_2 + j\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}) + R + m)\psi$$

$$= \quad \mathcal{B}^2 e^{-2\mathcal{G}}(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\psi \tag{C.7}$$

in which $R = C + D + E$ does not involve $\mathbf{p}$ terms(thus we have Eq(3.190) in text). Using the exponential form in Eq.(C.2) we can find

$$A = e^F chK$$

$$B = e^F shK$$

Now we multiply both sides of Eq.(C.7) by

$$(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})^{-1} = \frac{(A - B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})}{A^2 - B^2} \tag{C.8}$$

and find

$$\begin{aligned} & (A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})^{-1}[-2i(A' + B'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})]\hat{\mathbf{r}} \cdot \mathbf{p} \\ = & \quad -2i[\frac{AA' - BB' + (AB' - A'B)\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}}{A^2 - B^2}]\hat{\mathbf{r}} \cdot \mathbf{p} \\ = & \quad -2i(F' + K'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} \cdot \mathbf{p}, \end{aligned} \tag{C.9}$$

$$\begin{aligned} & (A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})^{-1} i\frac{2B}{r}[2\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p} - (\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p})] \\ = & \quad \frac{2i\sinh(K)\cosh(K)}{r}[2\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p} - (\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p})] + G \end{aligned} \tag{C.10}$$

where

$$
\begin{aligned}
G &= -\frac{2i\sinh(K)\sinh(K)}{r}\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}[2\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}\,\hat{\mathbf{r}}\cdot\mathbf{p}-(\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\mathbf{p}+\sigma_2\cdot\hat{\mathbf{r}}\sigma_1\cdot\mathbf{p})] \\
&= -\frac{2i\sinh^2(K)}{r}[2\,\hat{\mathbf{r}}\cdot\mathbf{p}-(\sigma_2\cdot\hat{\mathbf{r}}\sigma_2\cdot\mathbf{p}+\sigma_1\cdot\hat{\mathbf{r}}\sigma_1\cdot\mathbf{p})] \\
&= -\frac{2i\sinh^2(K)}{r}[2\,\hat{\mathbf{r}}\cdot\mathbf{p}-(\hat{\mathbf{r}}\cdot\mathbf{p}+i\sigma_2\cdot(\hat{\mathbf{r}}\times\mathbf{p})+\hat{\mathbf{r}}\cdot\mathbf{p}+i\sigma_1\cdot(\hat{\mathbf{r}}\times\mathbf{p}))] \\
&= -\frac{2i\sinh^2(K)}{r}[-i(\sigma_1+\sigma_2)\cdot(\hat{\mathbf{r}}\times\mathbf{p}))] \\
&= -\frac{2\sinh^2(K)}{r^2}\vec{L}\cdot(\sigma_1+\sigma_2) \qquad \text{(C.11)}
\end{aligned}
$$

$$
\begin{aligned}
&(A+B\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})^{-1}\frac{g'}{2r}B[2\sigma_1\cdot\sigma_2-4ir\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}\,\hat{\mathbf{r}}\cdot\mathbf{p} \\
&+2ir(\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\mathbf{p}+\sigma_2\cdot\hat{\mathbf{r}}\sigma_1\cdot\mathbf{p})-6\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}] \\
= \;&\frac{ig'\sinh(K)\cosh(K)}{2r}[-4r\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}\,\hat{\mathbf{r}}\cdot\mathbf{p}+2r(\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\mathbf{p}+\sigma_2\cdot\hat{\mathbf{r}}\sigma_1\cdot\mathbf{p}) \\
&-2i\sigma_1\cdot\sigma_2+6i\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}]+H \qquad \text{(C.12)}
\end{aligned}
$$

where

$$
\begin{aligned}
H &= -\frac{ig'\sinh^2(K)}{2r}\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}[-4r\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}\,\hat{\mathbf{r}}\cdot\mathbf{p}+2r(\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\mathbf{p}+\sigma_2\cdot\hat{\mathbf{r}}\sigma_1\cdot\mathbf{p}) \\
&\quad -2i\sigma_1\cdot\sigma_2+6i\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}] \\
&= -\frac{ig'\sinh^2(K)}{2r}[-4r\,\hat{\mathbf{r}}\cdot\mathbf{p}+2r(\sigma_1\cdot\hat{\mathbf{r}}\sigma_1\cdot\mathbf{p}+\sigma_2\cdot\hat{\mathbf{r}}\sigma_2\cdot\mathbf{p})-2i\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}\sigma_1\cdot\sigma_2+6i] \\
&= -\frac{ig'\sinh^2(K)}{2r}[-4r\,\hat{\mathbf{r}}\cdot\mathbf{p}+2r(\hat{\mathbf{r}}\cdot\mathbf{p}+i\sigma_2\cdot(\hat{\mathbf{r}}\times\mathbf{p})+\hat{\mathbf{r}}\cdot\mathbf{p}+i\sigma_1\cdot(\hat{\mathbf{r}}\times\mathbf{p})) \\
&\quad -2i\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}\sigma_1\cdot\sigma_2+6i] \\
&= -\frac{ig'\sinh^2(K)}{2r}[2i(\sigma_1+\sigma_2)\cdot L-2i(1-\sigma_1\cdot\sigma_2+\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})+6i] \\
&= \frac{g'\sinh^2(K)}{2r}[2(\sigma_1+\sigma_2)\cdot L-2(1-\sigma_1\cdot\sigma_2+\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})+6] \\
&= \frac{g'\sinh^2(K)}{2r}[2\vec{L}\cdot(\sigma_1+\sigma_2)-2\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}+2\sigma_1\cdot\sigma_2+4] \qquad \text{(C.13)}
\end{aligned}
$$

in which G and H do not contain linear $\mathbf{p}$ type of terms. Now collect the three different linear $\mathbf{p}$ type of terms in Eq(C.7):

$$(-2iF' - ig')\hat{\mathbf{r}} \cdot \mathbf{p}, \tag{C.14}$$

$$(-2i\frac{\sinh(K)\cosh(K)}{r} - ih' + ig'\sinh(K)\cosh(K))(\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{\mathbf{r}}\sigma_1 \cdot \mathbf{p}), \tag{C.15}$$

$$(4i\frac{\sinh(K)\cosh(K)}{r} - 2i\sinh(K)\cosh(K)g' - 2iK')\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p}. \tag{C.16}$$

If we set the first equation to 0, we obtain the expected result

$$F' = -g'/2. \tag{C.17}$$

if we set $h' = -K'$ and use $\mathbf{p} = \hat{\mathbf{r}}(\hat{\mathbf{r}}.\mathbf{p}) - \frac{\hat{\mathbf{r}} \times L}{r}$ to combine the two expressions(C.15 and C.16), we get

$$(2\frac{\sinh(K)\cosh(K)}{r} + h' - g'\sinh(K)\cosh(K))\frac{\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\vec{L} \cdot (\sigma_1 + \sigma_2)}{r} \tag{C.18}$$

which contain no $\hat{\mathbf{r}} \cdot \mathbf{p}$. Thus the scale change

$$\Psi = \exp(-g/2)\exp(-h\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\psi \tag{C.19}$$

eliminates the linear $\mathbf{p}$ terms.

Further note that

$$(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})^{-1}(k\sigma_1 \cdot \sigma_2 + j\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})$$

$$= (k\sigma_1 \cdot \sigma_2 + j\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})$$

and

$$(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})^{-1}C\psi = -g'(F' + K'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})\psi \tag{C.20}$$

$$(A + B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})^{-1}D\psi =$$

$$= \frac{(A - B\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})}{A^2 - B^2}\{-2h'[\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}A' + B'] - \frac{2h'B}{r}[2 - \sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}} + \sigma_1 \cdot \sigma_2 + \vec{L} \cdot (\sigma_1 + \sigma_2)]\}\psi$$

$$= \{-2h'\frac{(AB' + (AA' - BB')\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}} - BA')}{A^2 - B^2}$$

$$-2h'\frac{\cosh(K)\sinh(K)}{r}[\vec{L} \cdot (\sigma_1 + \sigma_2) + 2 - \sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}} + \sigma_1 \cdot \sigma_2]$$

$$+2h'\frac{\sinh^2(K)}{r}[\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\vec{L} \cdot (\sigma_1 + \sigma_2) + 3\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}} - \sigma_1 \cdot \sigma_2]\}\psi$$

$$= -2h'(K'+F'\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})\psi - 2h'\frac{\cosh(K)\sinh(K)}{r}[\vec{L}\cdot(\sigma_1+\sigma_2)+2-\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}+\sigma_1\cdot\sigma_2]\psi$$

$$+2h'\frac{\sinh^2(K)}{r}[\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}\vec{L}\cdot(\sigma_1+\sigma_2)+3\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}-\sigma_1\cdot\sigma_2]\psi$$

$$(A+B\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})^{-1}E\psi =$$

$$= \frac{(A-B\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})}{A^2-B^2}\{-(A''+B''\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})-\frac{2}{r}(A'+B'\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})-\frac{B}{r^2}(2\sigma_1\cdot\sigma_2-6\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})\}\psi$$

$$= \{-\frac{(AA''-BB''+(AB''-BA'')\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})}{A^2-B^2}-\frac{2}{r}\frac{(AA'-BB'+(AB'-BA')\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})}{A^2-B^2}$$

$$-\frac{2\cosh(K)\sinh(K)(\sigma_1\cdot\sigma_2-3\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})}{r^2}+2\frac{\sinh^2(K)}{r^2}(\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}-\sigma_1\cdot\sigma_2-2)\}\psi$$

$$= \{-[F''+F'^2+K'^2+(2F'K'+K'')\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}]-\frac{2}{r}[F'+K'\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}]$$

$$-2\frac{\cosh(K)\sinh(K)}{r^2}(\sigma_1\cdot\sigma_2-3\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})+2\frac{\sinh^2(K)}{r^2}(\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}-\sigma_1\cdot\sigma_2-2)\}\psi \quad \text{(C.21)}$$

So combining all terms, we have

$$\{\mathbf{p}^2+\frac{g'}{2r}\vec{L}\cdot(\sigma_1+\sigma_2)+k\sigma_1\cdot\sigma_2+j\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}$$

$$+(2\frac{\sinh(K)\cosh(K)}{r}+h'-g'\sinh(K)\cosh(K))\frac{\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}\vec{L}\cdot(\sigma_1+\sigma_2)}{r}$$

$$+\frac{g'\cosh(K)\sinh(K)}{r}(\sigma_1\cdot\sigma-3\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})$$

$$+\frac{g'\sinh^2(K)}{2r}[2\vec{L}\cdot(\sigma_1+\sigma_2)-2\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}+2\sigma_1\cdot\sigma_2+4]$$

$$-2\frac{\sinh^2(K)}{r^2}\vec{L}\cdot(\sigma_1+\sigma_2)-g'(F'+K'\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})-2h'(K'+F'\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})$$

$$-2h'\frac{\cosh(K)\sinh(K)}{r}[\vec{L}\cdot(\sigma_1+\sigma_2)+2-\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}+\sigma_1\cdot\sigma_2]$$

$$+2h'\frac{\sinh^2(K)}{r}[\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}\vec{L}\cdot(\sigma_1+\sigma_2)+3\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}-\sigma_1\cdot\sigma_2]$$

$$-[F'' + F'^2 + K'^2 + (2F'K' + K'')\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}] - \frac{2}{r}[F' + K'\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}]$$

$$-2\frac{\cosh(K)\sinh(K)}{r^2}(\sigma_1 \cdot \sigma_2 - 3\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}})$$

$$+2\frac{\sinh^2(K)}{r^2}(\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}} - \sigma_1 \cdot \sigma_2 - 2) + m\}\psi$$

$$= \mathcal{B}^2 e^{-2\mathcal{G}}\psi \tag{C.22}$$

Grouping the above equation by $\mathbf{p}^2$ term , spin-orbit angular momentum term $L \cdot (\sigma_1 + \sigma_2)$, spin-spin term $(\sigma_1 \cdot \sigma_2)$, tensor term $(\sigma_1 \cdot \hat{\mathbf{r}})(\sigma_2 \cdot \hat{\mathbf{r}})$, additional spin dependent term $(\sigma_1 \cdot \hat{\mathbf{r}})(\sigma_2 {\cdot} p) + (\sigma_2 \cdot \hat{\mathbf{r}})(\sigma_1 {\cdot} p)$ and spin independent terms, above equation becomes

$$\{\mathbf{p}^2 + \frac{2g'\sinh^2(K)}{r} - g'F' - 2h'K' - \frac{4h'\cosh(K)\sinh(K)}{r}$$

$$-F'' - F'^2 - K'^2 - \frac{2}{r}F' - \frac{4\sinh^2(K)}{r^2}$$

$$+\vec{L} \cdot (\sigma_1 + \sigma_2)[\frac{g'}{2r} + \frac{g'\sinh^2(K)}{r} - \frac{2\sinh^2(K)}{r^2} - 2h'\frac{\cosh(K)\sinh(K)}{r}]$$

$$+\sigma_1 \cdot \hat{\mathbf{r}}\sigma_2 \cdot \hat{\mathbf{r}}\vec{L} \cdot (\sigma_1 + \sigma_2)(2h'\frac{\sinh^2 K}{r} + 2\frac{\sinh K\cosh K}{r^2} + \frac{h'}{r} - \frac{g'\sinh K\cosh K}{r})$$

$$+\sigma_1\cdot\sigma_2[k+\frac{g'\cosh(K)\sinh(K)}{r}+\frac{g'\sinh^2(K)}{r}-2h'\frac{\cosh(K)\sinh(K)}{r}$$

$$-2h'\frac{\sinh^2(K)}{r}-2\frac{\cosh(K)\sinh(K)}{r^2}-2\frac{\sinh^2(K)}{r^2}]$$

$$+\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}[j-\frac{3g'\cosh K\sinh K}{r}-\frac{g'\sinh^2 K}{r}-g'K'-2h'F'+2h'\frac{\cosh K\sinh K}{r}$$

$$+6h'\frac{\sinh^2(K)}{r}-(2F'K'+K'')-\frac{2}{r}K'+6\frac{\cosh(K)\sinh(K)}{r^2}+2\frac{\sinh^2(K)}{r^2}]+m\}\psi$$

$$=\ \mathcal{B}^2e^{-2\mathcal{G}}\psi \tag{C.23}$$

C.2 Derivation Of Radial Equations

The following are radial eigenvalue equations after getting rid of the first derivative terms for singlet states$^1S_0$, $^1P_1$, $^1D_2$(a general singlet ^1J_j), triplet states $^3P_1$(a general let ^3J_j), and a general $s=1$, $j=l+1$ ($^3P_0$,$^3S_1$ states), a general $s=1$, $j=l+1$ ($^3D_1$ state)

Compare Eq.(C.1) with Eq.(3.175) we can find

$$g'=2\mathcal{G}'-\frac{E_2M_2+M_1E_1}{\mathcal{D}}(J+L)'=2\mathcal{G}'-\ln'\mathcal{D}, \tag{C.24}$$

$$h' = \frac{(J-L)'}{2}, \tag{C.25}$$

$$k = \frac{1}{2}\nabla^2\mathcal{G}+\frac{1}{2}\mathcal{G}'^2 - \frac{1}{2}\mathcal{G}'\ln'\mathcal{D} - \frac{1}{2}\mathcal{G}'C' - \frac{1}{2}\frac{\mathcal{G}'}{r} - \frac{1}{2}\frac{(-C+J-L)'}{r}, \tag{C.26}$$

$$j = -\frac{1}{2}\nabla^2(-C+J-L) - \frac{1}{2}\nabla^2\mathcal{G} - \mathcal{G}'(-C+J-L)' - \mathcal{G}'^2 + \frac{3}{2r}\mathcal{G}'$$

$$+\frac{3}{2r}(-C+J-L)' + \frac{1}{2}\ln'\mathcal{D}(\mathcal{G}-C+J-L)', \tag{C.27}$$

$$m = -\frac{1}{2}\nabla^2\mathcal{G}-\frac{1}{4}\mathcal{G}'^2 - \frac{1}{4}(C+J-L)'(-C+J-L)' + \frac{1}{2}\mathcal{G}'\ln'\mathcal{D}. \tag{C.28}$$

$^1S_0$, $^1P_1$, $^1D_2$ (a general singlet ^1J_j)

$\vec{\mathbf{L}}\cdot(\sigma_1+\sigma_2) = 0$, $\sigma_1\cdot\sigma_2 = -3$, $\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}} = -1$. Making use of $F' = -\frac{g'}{2}$ and $K' = -h'$, Eq.(C.23) becomes

$$\{-\frac{d^2}{dr^2}+\frac{j(j+1)}{r^2}+\frac{2g'\sinh^2 h}{r}+\frac{g'^2}{2}+2h'^2+4h'\frac{\cosh h\sinh h}{r}+\frac{g''}{2}-\frac{g'^2}{4}-h'^2+\frac{g'}{r}-\frac{4\sinh^2 h}{r^2}$$

$$-3[k-\frac{g'\cosh h\sinh h}{r}+\frac{g'\sinh^2 h}{r}+2h'\frac{\cosh h\sinh h}{r}-2h'\frac{\sinh^2 h}{r}+2\frac{\cosh h\sinh h}{r^2}-2\frac{\sinh^2 h}{r^2}]$$

$$-[j+\frac{3g'\cosh h\sinh h}{r}-\frac{g'\sinh^2 h}{r}+g'h'+h'g'-2h'\frac{\cosh h\sinh h}{r}+6h'\frac{\sinh^2 h}{r}$$

$$-(h'g'-h'')+\frac{2h'}{r}-6\frac{\cosh h\sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}]+m\}v=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}v,$$

Simplifying above equation, we get

$$\{-\frac{d^2}{dr^2}+\frac{j(j+1)}{r^2}+\frac{g'^2}{4}+h'^2+\frac{g''}{2}+\frac{g'}{r}-3k-j-g'h'-h''-\frac{2h'}{r}+m\}v=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}v,$$

Substituting g', h', k, j and m to above equation, we finally find

$$\{-\frac{d^2}{dr^2}+\frac{j(j+1)}{r^2}+\frac{(2\mathcal{G}-\ln(\mathcal{D})-J+L)'^2}{4}+\frac{(2\mathcal{G}-\ln(\mathcal{D})-J+L)''}{2}+\frac{(2\mathcal{G}-\ln(\mathcal{D})-J+L)'}{r}$$

$$+\frac{1}{2}\nabla^2(-C+J-L-3\mathcal{G})-\frac{1}{4}(C+J-L-\mathcal{G}+2\ln(\mathcal{D}))'(-C+J-L-3\mathcal{G})'\}v=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}v,$$

$^3P_1$(a general triplet ^3J_j)

$\vec{L}\cdot(\sigma_1+\sigma_2)=-2$, $\sigma_1\cdot\sigma_2=1$, $\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}=$1. Making use of $F'=-\frac{g'}{2}$ and $K'=-h'$, Eq.(C.23) becomes

$$\{-\frac{d^2}{dr^2}+\frac{j(j+1)}{r^2}+\frac{2g'\sinh^2 h}{r}+\frac{g'^2}{2}+2h'^2+4h'\frac{\cosh h\sinh h}{r}+\frac{g''}{2}-\frac{g'^2}{4}-h'^2+\frac{g'}{r}-\frac{4\sinh^2 h}{r^2}$$

$$-2[\frac{g'}{2r}+\frac{g'\sinh^2 h}{r}-2\frac{\sinh^2 h}{r^2}+2h'\frac{\cosh h\sinh h}{r}]$$

$$-2[2h'\frac{\sinh^2 h}{r}-2\frac{\cosh h\sinh h}{r^2}+\frac{h'}{r}+\frac{g'\cosh h\sinh h}{r}]$$

$$+[k-\frac{g'\cosh h\sinh h}{r}+\frac{g'\sinh^2 h}{r}+2h'\frac{\cosh h\sinh h}{r}-2h'\frac{\sinh^2 h}{r}+2\frac{\cosh h\sinh h}{r^2}-2\frac{\sinh^2 h}{r^2}]$$

$$[j+\frac{3g'\cosh h\sinh h}{r}-\frac{g'\sinh^2 h}{r}+g'h'+h'g'-2h'\frac{\cosh h\sinh h}{r}+6h'\frac{\sinh^2 h}{r}$$

$$-(h'g'-h'')+\frac{2h'}{r}-6\frac{\cosh h\sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}]+m\}v=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}v,$$

Simplifying above equation, we get

$$\{-\frac{d^2}{dr^2}+\frac{j(j+1)}{r^2}+\frac{g'^2}{4}+h'^2+\frac{g''}{2}+k+j+g'h'+h''+m\}v=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}v,$$

Substituting g', h', k, j and m to above equation, we finally find

$$\{-\frac{d^2}{dr^2}+\frac{2}{r^2}+\frac{(2\mathcal{G}-\ln(\mathcal{D})+J-L)'^2}{4}+\frac{(2\mathcal{G}-\ln(\mathcal{D})+J-L)''}{2}+\frac{(\mathcal{G}+J-L-C)'}{r}$$

$$-\frac{1}{2}\nabla^2(-C+J-L+\mathcal{G})+\frac{1}{4}(2\ln(\mathcal{D})-(C+J-L+3\mathcal{G}))'(J-L-C+\mathcal{G})'\}v=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}v,$$

$s=1,\ j=l+1$ ($^3S_1$ states)

$\vec{L}\cdot(\sigma_1+\sigma_2)=2(j-1)$, $\sigma_1\cdot\sigma_2=1$, $\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}=\frac{1}{2j+1}$ (diagonal term), and $\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}=\frac{2\sqrt{j(j+1)}}{2j+1}$(off diagonal term). Making use of $F'=-\frac{g'}{2}$ and $K'=-h'$, Eq.(C.23) becomes

$$\{-\frac{d^2}{dr^2}+\frac{j(j-1)}{r^2}+\frac{2g'\sinh^2 h}{r}+\frac{g'^2}{2}+2h'^2+4h'\frac{\cosh h\sinh h}{r}+\frac{g''}{2}-\frac{g'^2}{4}-h'^2+\frac{g'}{r}-\frac{4\sinh^2 h}{r^2}$$

$$+2(j-1)[\frac{g'}{2r}+\frac{g'\sinh^2 h}{r}-2\frac{\sinh^2 h}{r^2}+2h'\frac{\cosh h\sinh h}{r}]$$

$$+\frac{2(j-1)}{2j+1}[2h'\frac{\sinh^2 h}{r}-2\frac{\cosh h\sinh h}{r^2}+\frac{h'}{r}+\frac{g'\cosh h\sinh h}{r}]$$

$$+[k-\frac{g'\cosh h\sinh h}{r}+\frac{g'\sinh^2 h}{r}+2h'\frac{\cosh h\sinh h}{r}-2h'\frac{\sinh^2 h}{r}+2\frac{\cosh h\sinh h}{r^2}-2\frac{\sinh^2 h}{r^2}]$$

$$+\frac{1}{2j+1}[j+\frac{3g'\cosh h\sinh h}{r}-\frac{g'\sinh^2 h}{r}+g'h'+h'g'-2h'\frac{\cosh h\sinh h}{r}+6h'\frac{\sinh^2 h}{r}$$

$$-(h'g'-h'')+\frac{2h'}{r}-6\frac{\cosh h\sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}]+m\}u_+$$

$$+\frac{2\sqrt{j(j+1)}}{2j+1}\{j+\frac{3g'\cosh h\sinh h}{r}-\frac{g'\sinh^2 h}{r}+g'h'+h'g'-2h'\frac{\cosh h\sinh h}{r}+6h'\frac{\sinh^2 h}{r}$$

$$-(h'g'-h'')+\frac{2h'}{r}-6\frac{\cosh h\sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}$$

$$+2(j-1)[\frac{2h'\sinh^2(h)}{r}-\frac{2\cosh(h)\sinh(h)}{r^2}+\frac{h'}{r}+\frac{g'\cosh(h)\sinh(h)}{r}]\}u_-=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}u_+,$$

simplifying above equation, we get

$$\{-\frac{d^2}{dr^2}+\frac{j(j-1)}{r^2}+\frac{3g'\sinh^2 h}{r}+6h'\frac{\cosh h\sinh h}{r}-\frac{6\sinh^2 h}{r^2}-\frac{g'\cosh h\sinh h}{r}$$

$$-2h'\frac{\sinh^2 h}{r}+2\frac{\cosh h\sinh h}{r^2}+\frac{g'^2}{4}+h'^2+\frac{g''}{2}+\frac{g'}{r}+k$$

$$+2(j-1)[\frac{g'}{2r}+\frac{g'\sinh^2 h}{r}-2\frac{\sinh^2 h}{r^2}+2h'\frac{\cosh h\sinh h}{r}]$$

$$+\frac{2(j-1)}{2j+1}[2h'\frac{\sinh^2 h}{r}-2\frac{\cosh h\sinh h}{r^2}+\frac{h'}{r}+\frac{g'\cosh h\sinh h}{r}]$$

$$+\frac{1}{2j+1}[\frac{3g'\cosh h\sinh h}{r}-\frac{g'\sinh^2 h}{r}-2h'\frac{\cosh h\sinh h}{r}+6h'\frac{\sinh^2 h}{r}$$

$$-6\frac{\cosh h\sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}+j+g'h'+h''+\frac{2h'}{r}]+m\}u_+$$

$$+\frac{2\sqrt{j(j+1)}}{2j+1}\{\frac{3g'\cosh h\sinh h}{r}-\frac{g'\sinh^2 h}{r}-2h'\frac{\cosh h\sinh h}{r}+6h'\frac{\sinh^2 h}{r}$$

$$-6\frac{\cosh h\sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}+j+g'h'+h''+\frac{2h'}{r}$$

$$+2(j-1)[\frac{2h'\sinh^2(h)}{r}-\frac{2\cosh(h)\sinh(h)}{r^2}+\frac{h'}{r}+\frac{g'\cosh(h)\sinh(h)}{r}]\}u_-=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}u_+,$$

$s=1,\ j=l-1$ (${}^3P_0,{}^3D_1$ states)

$\vec{L}\cdot(\sigma_1+\sigma_2)=-2(j+2),\ \sigma_1\cdot\sigma_2=1,\ \sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}=-\frac{1}{2j+1}$(diagonal term), and $\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}=\frac{2\sqrt{j(j+1)}}{2j+1}$(off diagonal term). Making use of $F'=-\frac{g'}{2}$ and $K'=-h'$, Eq.(C.23) becomes

$$\{-\frac{d^2}{dr^2}+\frac{(j+1)(j+2)}{r^2}+\frac{2g'\sinh^2 h}{r}+\frac{g'^2}{2}+2h'^2+4h'\frac{\cosh h\sinh h}{r}+\frac{g''}{2}-\frac{g'^2}{4}-h'^2+\frac{g'}{r}-\frac{4\sinh^2 h}{r^2}$$

$$-2(j+2)[\frac{g'}{2r}+\frac{g'\sinh^2 h}{r}-2\frac{\sinh^2 h}{r^2}+2h'\frac{\cosh h\sinh h}{r}]$$

$$+\frac{2(j+2)}{2j+1}[2h'\frac{\sinh^2 h}{r}-2\frac{\cosh h\sinh h}{r^2}+\frac{h'}{r}+\frac{g'\cosh h\sinh h}{r}]$$

$$+[k-\frac{g'\cosh h\sinh h}{r}+\frac{g'\sinh^2 h}{r}+2h'\frac{\cosh h\sinh h}{r}-2h'\frac{\sinh^2 h}{r}+2\frac{\cosh h\sinh h}{r^2}-2\frac{\sinh^2 h}{r^2}]$$

$$-\frac{1}{2j+1}[j+\frac{3g'\cosh h\sinh h}{r}-\frac{g'\sinh^2 h}{r}+g'h'+h'g'-2h'\frac{\cosh h\sinh h}{r}+6h'\frac{\sinh^2 h}{r}$$

$$-(h'g'-h'')+\frac{2h'}{r}-6\frac{\cosh h\sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}]+m\}u_-$$

$$+\frac{2\sqrt{j(j+1)}}{2j+1}\{j+\frac{3g'\cosh h\sinh h}{r}-\frac{g'\sinh^2 h}{r}+g'h'+h'g'-2h'\frac{\cosh h\sinh h}{r}+6h'\frac{\sinh^2 h}{r}$$

$$-(h'g'-h'')+\frac{2h'}{r}-6\frac{\cosh h\sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}$$

$$-2(j+2)[\frac{2h'\sinh^2(h)}{r}-\frac{2\cosh(h)\sinh(h)}{r^2}+\frac{h'}{r}+\frac{g'\cosh(h)\sinh(h)}{r}]\}u_+=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}u_-,$$

simplifying above equation, we get

$$\{-\frac{d^2}{dr^2}+\frac{(j+1)(j+2)}{r^2}+\frac{3g'\sinh^2 h}{r}+6h'\frac{\cosh h\sinh h}{r}-\frac{6\sinh^2 h}{r^2}-\frac{g'\cosh h\sinh h}{r}$$

$$-2h'\frac{\sinh^2 h}{r}+2\frac{\cosh h\sinh h}{r^2}+\frac{g'^2}{4}+h'^2+\frac{g''}{2}+\frac{g'}{r}+k$$

$$+2(j+2)[\frac{g'}{2r}+\frac{g'\sinh^2 h}{r}-2\frac{\sinh^2 h}{r^2}+2h'\frac{\cosh h\sinh h}{r}]$$

$$+\frac{2(j-1)}{2j+1}[2h'\frac{\sinh^2 h}{r}-2\frac{\cosh h\sinh h}{r^2}+\frac{h'}{r}+\frac{g'\cosh h\sinh h}{r}]$$

$$-\frac{1}{2j+1}[\frac{3g'\cosh h\sinh h}{r}-\frac{g'\sinh^2 h}{r}-2h'\frac{\cosh h\sinh h}{r}+6h'\frac{\sinh^2 h}{r}$$

$$-6\frac{\cosh h\sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}+j+g'h'+h''+\frac{2h'}{r}]+m\}u_-$$

$$+\frac{2\sqrt{j(j+1)}}{2j+1}\{\frac{3g'\cosh h\sinh h}{r}-\frac{g'\sinh^2 h}{r}-2h'\frac{\cosh h\sinh h}{r}+6h'\frac{\sinh^2 h}{r}$$

$$-6\frac{\cosh h\sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}+j+g'h'+h''+\frac{2h'}{r}$$

$$-2(j+2)[\frac{2h'\sinh^2(h)}{r}-\frac{2\cosh(h)\sinh(h)}{r^2}+\frac{h'}{r}+\frac{g'\cosh(h)\sinh(h)}{r}]\}u_+=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}u_-,$$

in above equation u_+ and u_- are the wavefunction for states $s=1$, $j=l+1$ and $s=1$, $j=l-1$. Here the $h=\frac{J-L}{2}$ and $g=2\mathcal{G}-\ln(\mathcal{D})$, and note that g and $\mathcal{G}$ are different variables. Above two equation for states $s=1$, $j=l+1$ and $s=1$, $j=l-1$ are coupled equations, we can put the two equations in the following form

$$\{-\frac{d^2}{dr^2}+\Phi_{11}(r)\}u_++\Phi_{12}(r)u_-=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}u_+$$

$$\{-\frac{d^2}{dr^2}+\frac{6}{r^2}+\Phi_{22}(r)\}u_-+\Phi_{21}(r)u_+=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}u_-$$

where

$$\Phi_{11}(r) = \{\frac{3g'\sinh^2 h}{r} + 6h'\frac{\cosh h \sinh h}{r} - \frac{6\sinh^2 h}{r^2} - \frac{g'\cosh h \sinh h}{r}$$

$$-2h'\frac{\sinh^2 h}{r} + 2\frac{\cosh h \sinh h}{r^2} + \frac{g'^2}{4} + h'^2 + \frac{g''}{2} + \frac{g'}{r} + k$$

$$+2(j-1)[\frac{g'}{2r} + \frac{g'\sinh^2 h}{r} - 2\frac{\sinh^2 h}{r^2} + 2h'\frac{\cosh h \sinh h}{r}]$$

$$+\frac{2(j-1)}{2j+1}[2h'\frac{\sinh^2 h}{r} - 2\frac{\cosh h \sinh h}{r^2} + \frac{h'}{r} + \frac{g'\cosh h \sinh h}{r}]$$

$$+\frac{1}{2j+1}[\frac{3g'\cosh h \sinh h}{r} - \frac{g'\sinh^2 h}{r} - 2h'\frac{\cosh h \sinh h}{r} + 6h'\frac{\sinh^2 h}{r}$$

$$-6\frac{\cosh h \sinh h}{r^2} + 2\frac{\sinh^2 h}{r^2} + j + g'h' + h'' + \frac{2h'}{r}] + m\} - \mathcal{B}^2 \mathrm{e}^{-2\mathcal{G}} + b^2(w)$$

$$\Phi_{12}(r) = \frac{2\sqrt{j(j+1)}}{2j+1}\{\frac{3g'\cosh h \sinh h}{r} - \frac{g'\sinh^2 h}{r} - 2h'\frac{\cosh h \sinh h}{r} + 6h'\frac{\sinh^2 h}{r}$$

$$-6\frac{\cosh h \sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}+j+g'h'+h''+\frac{2h'}{r},$$

$$+2(j-1)[\frac{2h'\sinh^2(h)}{r}-\frac{2\cosh(h)\sinh(h)}{r^2}+\frac{h'}{r}+\frac{g'\cosh(h)\sinh(h)}{r}]\}$$

$$\Phi_{22}(r)=\{\frac{3g'\sinh^2 h}{r}+6h'\frac{\cosh h \sinh h}{r}-\frac{6\sinh^2 h}{r^2}-\frac{g'\cosh h \sinh h}{r}$$

$$-2h'\frac{\sinh^2 h}{r}+2\frac{\cosh h \sinh h}{r^2}+\frac{g'^2}{4}+h'^2+\frac{g''}{2}+\frac{g'}{r}+k$$

$$+2(j+2)[\frac{g'}{2r}+\frac{g'\sinh^2 h}{r}-2\frac{\sinh^2 h}{r^2}+2h'\frac{\cosh h \sinh h}{r}]$$

$$+\frac{2(j-1)}{2j+1}[2h'\frac{\sinh^2 h}{r}-2\frac{\cosh h \sinh h}{r^2}+\frac{h'}{r}+\frac{g'\cosh h \sinh h}{r}]$$

$$-\frac{1}{2j+1}[\frac{3g'\cosh h \sinh h}{r}-\frac{g'\sinh^2 h}{r}-2h'\frac{\cosh h \sinh h}{r}+6h'\frac{\sinh^2 h}{r}$$

$$-6\frac{\cosh h\sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}+j+g'h'+h''+\frac{2h'}{r}]+m\}-\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}+b^2(w)$$

$$\Phi_{21}(r)=\frac{2\sqrt{j(j+1)}}{2j+1}\{\frac{3g'\cosh h\sinh h}{r}-\frac{g'\sinh^2 h}{r}-2h'\frac{\cosh h\sinh h}{r}+6h'\frac{\sinh^2 h}{r}$$

$$-6\frac{\cosh h\sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}+j+g'h'+h''+\frac{2h'}{r}$$

$$-2(j+2)[\frac{2h'\sinh^2(h)}{r}-\frac{2\cosh(h)\sinh(h)}{r^2}+\frac{h'}{r}+\frac{g'\cosh(h)\sinh(h)}{r}]\}$$

When $l=1$, $j=l-1=0$. Above coupled equation collapse to the uncoupled $^3P_0$ state. The eigenvalue equation for this $^3P_0$ state is

$^3P_0$

$$\{-\frac{d^2}{dr^2}+\frac{2}{r^2}+\frac{(2\mathcal{G}-\ln(\mathcal{D})-J+L)'^2}{4}+\frac{(2\mathcal{G}-\ln(\mathcal{D})-J+L)''}{2}+\frac{(\ln(\mathcal{D})-(4\mathcal{G}+J-L-2C))'}{r}$$

$$+\frac{1}{2}\nabla^2(-C+J-L+\mathcal{G})-\frac{1}{2}\mathcal{G}'C'+\frac{1}{4}(C^{'2}-(J-L)'^2)+\mathcal{G}'(\frac{5}{4}\mathcal{G}+J-L-C)'$$

$$-\frac{1}{2}\ln'(\mathcal{D})(J-L-C+\mathcal{G})'\}v=\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}v,$$

Our potential for above Schrödinger-like equation is

$$
\begin{aligned}
\Phi(r) \quad = \quad & \frac{(2\mathcal{G}-\ln(\mathcal{D})-J+L)'^2}{4}+\frac{(2\mathcal{G}-\ln(\mathcal{D})-J+L)''}{2}+\frac{(\ln(\mathcal{D})-(4\mathcal{G}+J-L-2C))'}{r} \\
& +\frac{1}{2}\nabla^2(-C+J-L+\mathcal{G})-\frac{1}{2}\mathcal{G}'C'+\frac{1}{4}(C^{'2}-(J-L)'^2)+\mathcal{G}'(\frac{5}{4}\mathcal{G}+J-L-C)' \\
& -\frac{1}{2}\ln'(\mathcal{D})(J-L-C+\mathcal{G})'-\mathcal{B}^2\mathrm{e}^{-2\mathcal{G}}+b^2(w),
\end{aligned}
$$

Right now, we can apply the techniques which already developed for the Schrödinger-like system in nonrelativistic quantum mechanics to above equations, we wish to compare this directly with Reid's potential and use it to fit the experimental phase shift data.

Vita

Bin Liu obtained his Master of Science in physics at University of Science and Technology of China(USTC) in 1992. His research area at UTSC was Large Scale Structure of the Universe. He studied the distribution of galaxies by using the methods of Monte Carlo simulations, correlation function and fractal dimension. From 1992 to 1996, He work as a research assistant at China Institute of Atomic Energy(CIAE, former Institute of Atomic Energy of Chinese Academy of Sciences) in Beijing, China. His research area at CIAE was Nuclear Structure. He studied the rare earth nuclei.

He came to University of Tennessee on August 1996 and finished his PhD in physics on August 2001. The topic of his dissertation was " Two Body Dirac Equations and Nucleon Nucleon Scattering Phase Shift Analysis ". He worked on the nucleon nucleon scattering phase shift analysis by using the two body Dirac equations of the constraint dynamics and the meson exchange model.